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ABSTRACT

This guide describes the content of a proposed mathematics course for prospective elementary school teachers. It is the result of a two-year study at Indiana University in which three existing courses were integrated and coordinated. For each unit of instruction, there are (1) remarks for motivation of study, (2) remarks on methods of teaching, and (3) instructions for making changes in the succession of topics covered in a text. [Not available in hardcopy due to marginal legibility of original document.] (RS)

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ONE-YEAR INTEGRATED MATHEMATICS
and
MATHEMATICS METHODS COURSE
for
PROSPECTIVE ELEMENTARY SCHOOL TEACHERS

by

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INDIANA PROJECT (CCSM)

PRELIMINARY MATERIALS

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RECOMMENDATIONS FROM THE EDC PROJECT

For the past two years, the EDC Project at Indiana University has involved members of the Mathematics Department and the School of Education in a coöperative study of the question: How should prospective elementary school teachers be prepared to teach mathematics to children?

During the summer of 1967, three members of the Mathematics Department and three members of the School of Education worked together to revise the existing mathematics and mathematics education curricula for elementary teachers. Every effort was made to correlate the contents of the two curricula. Unnecessary duplications of content and instructional materials were eliminated. Syllabi were prepared for the courses T104, T106, and E343, and texts and supplementary materials for these courses were chosen.

These syllabi and materials were used during the Spring semester of 1967-68. Matters were arranged so that two sections, made up of students who had taken T104 the preceding semester, were enrolled in both T106 and E343. For each of these sections, the two courses were taught in the same classroom during two adjacent class periods. One of the mathematics teachers evaluated the effectiveness of this program by comprehensive testing of these two sections. The other instructors involved in the program conducted informal evaluations with their sections. Each instructor was asked to suggest needed revisions in the prepared syllabi.

During the summer of 1968, a team constituted similarly to that of the previous summer revised the syllabi and textbook choices for T104, T106, and E343, for use in 1968-69. A syllabus was also prepared for an integrated mathematics course, described in Recommendation 1 below. Copies of these syllabi and related materials are appended to this report. In the recommendations the two parts of the integrated mathematics course just mentioned are denoted by "Int I" (5 credits) and "Int II" (3 credits).

On the basis of objective and subjective evidence gathered during the two years of the Project, the team members make the following recommendations:

1. Elementary school programs clearly reflect an integration of arithmetic, geometry, and algebra into a single course of study. The mathematics education course content reflects the elementary school curriculum. Yet in the three mathematics content courses T104 (arithmetic), T106 (geometry), and T108 (algebra), these three streams of thought are still separated. Thus it is RECOMMENDED that beginning in 1969-70 (the earliest practical date), the courses T104, T106, and T108 be replaced by the one-year, eight-credit, integrated mathematics course here called Int I — Int II (see syllabus, appended).

2. Evidence seems to point to the need for keeping the mathematics content courses as close as possible to the mathematics education course. It also seems clear that the content course should not be taken after the education course, although current policy allows this. In the existing Education program, the mathematics education course is taken just prior to student teaching. Thus it is RECOMMENDED that the mathematics content courses be removed from the 100 (Freshman) level and placed at the 200 or 300 level. It is further RECOMMENDED that T104 (or Int I) be made a prerequisite for T106 (or Int II) and E343, and that E343 be made a co-requisite for T106 (or Int II). Students should also be strongly urged to take T104 and T106 (or Int I and Int II) in adjacent school terms (i.e. either Fall-Spring, Spring-Summer, or Summer-Fall).

3. It is felt that additional attention should be given to the development of instructional materials to supplement the basic texts in the various courses. Members of the mathematics team have so far prepared two supplementary units for T104, covering material not included in the basic T104 text (copies of these units appended). Members of the mathematics education team have worked towards preparing packages of materials to supplement each unit of the E343 syllabus (some of this material is appended). It is RECOMMENDED that additional supplementary instructional materials be produced for use in both the mathematics and mathematics education programs. Additional monies and personnel will be needed to develop this aspect of the project further. For the most effective use of the materials developed for E343, it is RECOMMENDED that a learning laboratory be set up for E343 students, and that a diagnostic mathematics examination be given at the beginning of E343 to determine which students still need work on which topics.

4. It is felt that a student who receives a grade of D in T104 or T106 is not sufficiently prepared to teach mathematics in the elementary schools; yet it is possible, under current policies, for such students to attain certification. Thus it is RECOMMENDED that grades of C or above in T104 and T106 (or Int I and Int II) be required of all candidates for certification. Further, a grade of C or above in Int I should be a prerequisite for Int II. To make this requirement meaningful, it will be necessary to have more or less uniform grading standards for all sections of a given course. The Teaching Associates on the team, however, are opposed to the idea of giving all students the same examinations. Thus the team has prepared, for each mathematics course, a list of topics and skills which they feel a student should master in order to receive a grade of C or above; these lists are appended to the syllabi, and Teaching Associates will be encouraged to use them as standards.

5. An increasing (though still small) number of students entering the Education program have been so well prepared in high school that they already know the material covered in T104 and/or T106. Thus it is RECOMMENDED that a comprehensive mathematics examination, covering the material of T104 and T106, be offered during Registration Week each semester, and that students who score sufficiently high on this examination be given credit for T104 and/or T106 (or Int I and possibly Int II) and allowed (and encouraged) to take more advanced mathematics courses. (If Recommendation 4 is adopted, things should be arranged so that a student with a previous D grade in one of the courses can satisfy certification requirements and/or prerequisites for the next course by scoring sufficiently high on the examination.)

6. It is RECOMMENDED that the present policy of choosing T104 and T106 Teaching Associates mostly from the Mathematics Education program be continued, and maintained for Int I and Int II.

7. It is RECOMMENDED that the Teaching Associates for T104 and T106 (or Int I and Int II) be given considerably more supervision and advice than they presently receive. In particular:

a) The Teaching Associates for each course should be divided into groups of not more than six, each group teaching under the close supervision of a faculty member of the Mathematics Department. This is current policy, but since the faculty member must now perform his supervisory duties in addition to his full-time regular teaching job, supervision is not now adequate. Thus it is STRONGLY RECOMMENDED that sufficient faculty be hired to allow the supervision of six Teaching Associates to count as one-half of the faculty supervisor's total work load. Two additional faculty members should be given, as their total work load, the overall supervision of all the sections of T104 and T106 (or Int I and Int II), one course for each of them. A supervisor would visit at least one section meeting each day and would hold frequent conferences with his Teaching Associates, both individually and in groups, to discuss teaching problems and strategies, construction of tests, and so on.

b) The Teaching Associates in each course should be given desk space in the same office, and reference materials relevant to their course should be placed in that office.

c) All Teaching Associates should be given copies of the Teaching Associate's Handbook which will be produced by the staff of the Summer Seminar for Prospective Teaching Associates which was held in the Mathematics Department during the summer of 1968.

d) Teaching Associates should be encouraged to visit each other's classes.

e) Visual teaching aids, such as abaci, should be purchased and made available to the Teaching Associates. The Teaching Associates should also be encouraged to exploit the resources of the Audio-Visual Center.

COURSE OUTLINE FOR THE ONE-YEAR INTEGRATED MATHEMATICS COURSE FOR PROSPECTIVE ELEMENTARY SCHOOL TEACHERS

Main Textbook: Garstens and Jackson, Mathematics for Elementary School Teachers , Macmillan, 1967.

Supplementary Textbook:

Topics in Mathematics for Elementary School Teachers (Twenty-Ninth Yearbook), National Council of Teachers of Mathematics, Washington, D.C., 1964.

Teacher's References:

Peterson and Hashisaki, Theory of Arithmetic, 2nd ed., John Wiley and Sons, 1967.

Smeltzer, Man and Number, Collier Books, 1962.

Most of the topics to be stressed in each part of this course are excellently summarized in the final sections (entitled "Terminal Tasks") of each chapter of the main text, and in the Teacher's Manual accompanying the main text. Thus the lists of "Terminal Tasks" are to be used as the basic course outline. The following material is to be regarded as an amendment to the outline given by the "Terminal Tasks". For each unit of instruction, we here give: 1) remarks which can (and should) be made to motivate the study of the material, beyond those remarks made in the first section of each chapter of the main text (which should also be used); 2) remarks on methods of teaching the material, beyond those given in the Teacher's Manual; and 3) instructions to make various changes in the succession of topics covered in the text, either to include new topics or to omit given ones. Units of instruction not based on the text are described in full detail.

Unless specifically omitted below, all sections of the text are to be covered in this course, even those about which no remarks are made below about motivation or methods.

INTRODUCTION

Discuss goals of the course, method of grading, testing schedule, and so on. Learn students' names as quickly as possible. Have students introduce themselves to each other, and encourage them to work and study together outside of class. Good introductory films exist; one may be made available for showing to your class during this first week. Bring one or two sets of elementary-school mathematics textbooks to class and pass them around to let students see what they are going to be teaching.

The text will treat whole numbers as given, but the class should be given some motivational background. Develop the idea that a (whole) number is a concept associated with the size of a set of objects. At first numbers were not named, but only compared. A caveman might test whether he had more stone axes than his neighbor by pairing off the two sets of axes and seeing which set ran out first. A shepherd might put one pebble in his pocket for every sheep leaving the fold; later, he could tell if all had returned. Commerce eventually required men to name numbers larger than could be indicated by just holding up fingers. (E.g. "How many stones will be needed to build this pyramid?") How are we to name the various numbers? We could simply give separate, mutually unrelated names to each number; but this would be impractical. Get the class to think of grouping the set to be counted into groups of ten (or other small number), then grouping the groups of ten into groups of ten, etc. Study various numeration systems: Egyptian, Babylonian, Chinese, Mayan (see Smeltzer, pp. 29-55; Peterson & Hashisaki, Chap. 1). Make up a simple numeration system yourself, using funny symbols, and have the students try to decode it from five or six given examples.

Show the students how to write numbers in different bases (using Hindu-Arabic numerals), and have them translate some numbers from base ten to other bases and back again. (See 29th Yearbook, pp. 111-132.)

CHAPTER 2

We defer Chapter 1 until later since its material will be new and alien to the student, while that of Chapter 2 is familiar. Also, students should be more willing to study logic *per se* if the need for such a systematic study is previously impressed upon them by exposure to a few (mysterious) proofs. Some teachers may even wish to defer Chapter 1 until after Chapter 3.

The text treats addition and multiplication of whole numbers, and their basic properties, as known, and leaps directly into an axiomatic development. Again, additional background and motivation should be given. Define addition of whole numbers m and n as putting m objects together with n other objects and counting the resulting set. Define multiplication of m and n as putting together m different sets of n objects each and counting the resulting set. Point out that having done this once for each m and n, we can put down the results in a table which we can then memorize without further thought, but that the way we got the table should be kept in mind in case we forget part of the table. The set of whole numbers, together with the operations of addition and multiplication, form an example of a mathematical system. Go over the axioms of Sections 2.2--2.6 and convince the students of the truth of these axioms by means of experimentation and picture-drawing.

Motivate the axiomatic development of number systems as follows: We know two ways of convincing ourselves that a given true statement is true. We can verify it by experimentation, or we can deduce it logically from known facts. The first method is difficult, and not entirely convincing; the second method is comparatively easy and is completely convincing in the sense that the conclusions of a valid argument must be accepted if its premises are. So in developing a large collection of facts it makes sense to verify as few of them as possible by experiment and prove the others from these few; this is what we shall do.

Having made these remarks, do some of the proofs in the Exercises of Sections 2.2--2.6. Introduce the number line (whole numbers only) and view addition and (where defined) subtraction in terms of moving back and forth on the line. Cover Sections 2.8 and 2.9.

Now study the addition, subtraction, multiplication, and division algorithms (use the 29th Yearbook, pp. 133--166). It is crucial that elementary teachers know exactly why these computational methods work, and not think of them as magic rituals which will give the right answers if only one can remember how to perform them correctly. (The same goes for all other algorithms to be covered later, e.g. multiplication of fractions.) To test understanding of these algorithms, spend a day doing computations in other number bases. This will force the students to experience the same problems which their own students will later have with base-ten computation.

To motivate Section 2.10, present the whole numbers under multiplication as a structure in which the various numbers are "put together" from basic building blocks, namely the primes. State and illustrate the Fundamental Theorem of Arithmetic. Find some prime factorizations, g.c.d.'s, and l.c.m.'s. Give examples of systems in which factorizations into primes are *not* unique. Present some of the classical unsolved problems of number theory, e.g.: 1) Goldbach's Conjecture--Is every even number greater than 2 equal to a sum of two primes? 2) Twin Prime Problem--Are there an infinite number of twin primes (primes p such that $p+2$ is also prime)? 3) Fermat's Last "Theorem"--Does $x^n + y^n = z^n$ have any positive integral solutions x, y, and z if n is a whole number greater than 2? 4) Perfect Number Problem--Are there any *odd* perfect numbers (numbers equal to the sum of their proper divisors)?

Ask "How many primes are there?"; then prove Theorem 2.14, which does not answer this question but shows that the answer is not a finite number. Use this to lead into the question of how the sizes of infinite sets are to be compared. Propose the text's definition. Be prepared to counter the objection that since there are obviously more whole numbers than even whole numbers, the two sets cannot be the same size. Cover Section 2.12.

CHAPTER 1

Motivation. Having explored the territory by stumbling through a few proofs, we will now study proof techniques generally.

Methods. In Section 1.4, the text does not state the truth-value of $P \rightarrow Q$ when P and Q are statements and P is false; this should be done. To convince students that $P \rightarrow Q$ should be true when P is false, show by examples that in everyday speech "Either R or Q" means the same thing as "If not R, then Q". Then substitute "not P" for "R" and eliminate the double negation to see that "If P, then Q" means the smae thing as "Either (not P) or Q". The truth-values of $P \rightarrow Q$ can then be computed from knowledge of "or" and "not". *Stress* that the mere truth of $P \rightarrow Q$ does not imply a logical or causal connection between P and Q (the word "implication" in the text is misleading and should probably be replaced by "conditional").

CHAPTER 3

Motivation. So far we have applied mathematical methods only to the study of numbers. But there are other structures in everyday life which can be studied by mathematical methods.

Methods. Lead into Section 3.2 via clock arithmetic. Abstract the properties of a group from those of mod 3 arithmetic. Point out that anything we prove from the group axioms will be true of any structure which satisfies those axioms. Give an example of a non-numerical group, such as the group of rotations of a cube (use a real cube).

CHAPTER 4

Methods. Section 4.4 is just an example of a deductive system; the actual content of the theory there developed is of little importance. So do not stress this section too heavily. In Sections 4.6--4.7, get the students to come up with definitions of the various defined terms, based on their experience, and work with the class to get these definition into precise form (agreeing with those in the text). Do not go through all the proofs in detail; knowledge of the content of the content of the theorems is more important than memorizing proofs. An instructive practical exercise in set theory can be found in Peterson and Hashisaki (p. 55, #13).

CHAPTER 5

Methods. Try to get the students to come up with properties which the congruence correspondences should have; thus develop the axioms CS1--CS7. Go through the first proofs slowly, but speed up later. Again, the content of a theorem is more important than its proof. Students should learn from this course what a good proof is, and how to write simple ones, but they should not have to memorize hard proofs from the text. At the end of the chapter, briefly treat the reflection-rotation-translation approach to congruence. Use the article "Congruence Geometry for Junior High School Students" by Dennis and Sanders (Mathematics Teacher, April 1968). Review the classical compass-and-straightedge constructions, showing how they are consistent with the axioms of Chapter 5.

CHAPTER 6

Motivation. I had \$100 and I spent (on credit) \$200; how much money do I have now? I started three miles east of town and moved five miles westward; how far east of town am I now? I want to solve the equation $x+6=4$; what can I do? Answering these questions requires the creation of new numbers. What is the relationship these new numbers must hold to the known (whole) numbers? Trying to solve $x+6=4$ in whole numbers, I see that any whole number is too big to be x. Thus we want our new numbers to be less than zero. We want, e.g., $4-6$ to be equal to some number, but we must create a new number for this purpose. Why not just call this new number $(4-6)$? Plotting several such new numbers on an extension of the number line (e.g. $(4-6)$ goes 6 units left of 4), we see that we want, e.g., $(4-6)$ and $(5-7)$ to be the same number. How can we identify such numbers with each other in a neat, formal way? Having settled on the definitions of the new numbers, how shall we add and multiply them? How shall we relate them to the old numbers?

Methods. Make the above remarks at the beginning of the chapter, so that the students will know why we do what we later do.

CHAPTER 7

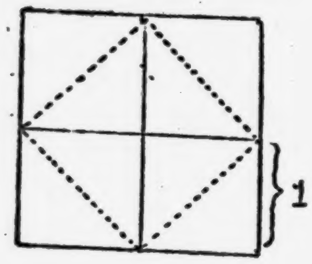
Motivation. I am counting the number of basketfuls of grain harvested on my farm and I come to a basket which is only partially full; how do I count it? I am measuring my field with a stick and I find that the field is more than 103 sticks long, but less than 104 sticks long; how long is my field? I want to solve the equation $3x=5$, but I see that 1 and all smaller integers are too small to be x, and 2 and all larger integers are too big; what can I do? We need to create more new numbers which lie in between the integers. We will do this in the same sort of way we created the integers.

Changes. After Section 7.7, spend a day doing story problems involving percentages, such as the students will later be teaching to their own students. Also teach "scientific notation" for writing rational numbers, e.g. 2.45×10^6, 1.97×10^{-5}. Section 7.8 may be omitted if time is short.

CHAPTER 8

Motiviation. In Section 8.6, after introducing infinite decimals ask: Why should we grant all infinite decimals the status of numbers? Reasons: 1) They form a nice ordered pattern in which we can embed the rational numbers such that the two orderings agree. 2) Under certain definitions of addition and multiplication of infinite decimals, not only do these operations agree with those of the rational numbers when the latter are embedded, but equations like $x^2 = 2$ now have solutions. But this equation did not have a solution in the rational numbers. (Prove this now.) 3) We want 2 to have a number as a square root, for later when we talk about measurement we will want a number for every distance measurable along the number line; and we can measure a distance whose square is 2.

Methods. Any proof that we can "measure a distance whose square is 2" will be a fudge at this point, since we have not yet studied area, the Pythagorean Theorem, etc. The simplest way is to argue that the dotted square in the figure must have area 2 since it is obviously half of the large square, which has area 4.



Additions. After Section 8.4, define commensurability and ask if any two line segments are commensurable. After showing $\sqrt{2}$ is irrational, return to this question and answer it.

CHAPTER 9

Changes. Study Appendix A right after Section 9.8.

CHAPTER 10

Motivation. For Section 10.2: Throughout mathematics, it has often been found that a non-numerical structure (in this case, the plane) can be more effectively studied if we associate numbers to its various points in some clever way. For Sections 10.3--10.4: Given two figures which have the same shape, what sort of transformation can we perform on it to get it to coincide with the other? We must expand or shrink it, and then move it rigidly (translate, rotate, or reflect). How can we rigorously describe "expanding" and "shrinking"?

Changes. Omit Sections 10.6 and 10.9. The interested student may wish to read the article "Rotations, Angles, and Trigonometry" by Troyer (Mathematics Teacher, Feb. 1968)

APPENDICES B--D

Methods. The teacher may assign the better students to lecture on these sections.

PROBABILITY AND STATISTICS

This topic is not covered in the text. Mimeographed text materials and teaching outlines will be distributed.

CALENDAR

To insure that each class covers all the material of this course, and that all the sections finish the first semester at the same point (so that students can change sections to accomodate their schedules), it is recommended that Teaching Associates follow this schedule:

FIRST SEMESTER (75 days of class)

Introduction	5 days
Chapter 2	15 days
Test	2 days
Chapter 1	5 days
Chapter 3	6 days
Test	2 days
Chapter 4	7 days
Chapter 5	11 days
Test	2 days
Chapter 6	9 days
Test	2 days
Chapter 7	7 days
Test and review	2 days

SECOND SEMESTER (45 days of class)

Chapter 8	11 days
Test	2 days
Chapter 9	10 days
Test	2 days
Chapter 10	6 days
Appendices B--D	4 days
Test	2 days
Probability/Statistics	5 days
Review (whole course)	3 days

Two days are allotted for tests so that the test may be discussed the day after it is given. It is also suggested that short quizzes (10-15 min.) be given frequently, as time permits.

COURSE OUTLINE - T104

Required Textbooks

P Peterson, J. A. and Hashisaki, J., Theory of Arithmetic, (second edition) John Wiley and Sons, Inc., New York, 1967.

Y National Council of Teachers of Mathematics, Enrichment Mathematics for the Grades, Twenty-Seventh Yearbook, National Council of Teachers of Mathematics, Washington, D. C., 1963.

Optional Textbooks (available in Swain Hall Library)

E Eves, Howard, An Introduction to the History of Mathematics, Holt, Rinehart, and Winston, New York, 1953.

M Meserve, B. E. and Sobel, M. A., Introduction to Mathematics, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1964.

MS Meserve, B. E. and Sobel, M. A., Elements of Mathematics, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1968.

R Rees, Paul K., Principles of Mahematics, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1959.

S Smeltzer, D., Man and Number, Collier Books, New York, 1962.

T Terry, G. S., Duodecimal Arithmetic, Longman, Green and Co., New York, 1938.

To the Student:

In recent years there has been a revolution in school mathematics. A quick glance into some newer elementary school mathematics texts will show that skill in computation is no longer sufficient criteria for teaching elementary school mathematics. The sequence of courses consisting of T-104, T-106, and T-108 is designed to give prospective elementary teachers the background in mathematics that they must have in order to teach mathematics successfully to elementary school pupils.

The courses T-104, T-106, and T-108 are concerned with mathematical ideas of two types: those which will be taught to elementary pupils and those which will give the prospective teacher a deeper understanding of the concepts he will teach. These courses are strictly mathematical in nature; a companion course, E-343, will instruct the prospective teacher in the techniques of the actual presentation of the mathematical ideas to his pupils.

In order that the students may benefit maximally from the lectures, it is strongly recommended they study beforehand the material presented in each lecture.

The texts required for T-104 are also used in T-106 and the students will probably have occasion to use them as personal references.

Course Content

1. History and Foundations
   P: 1.1 - 1.4, 1.5e, 1.5f
   M: 1.1
   E: selected topics
   S: selected topics
2. Logic
   M: 9.1 - 9.4
   Y: 282 - 290 (required reading)
   Y: 291 - 301
3. Numeration Systems
   P: 1.5a, b, c, d, 1.6
   P: Chapter 5 (complete)
   M: Chapter 2 (required reading)
   Y: pp. 41 - 49, pp. 234 - 239
   T: selected topics — Test 1
4. Set Theory and Relations
   P: Chapter 2 (complete)
   M: Chapter 4
   P: Chapter 3 (complete) — Test 2
5. Whole Numbers
   P: Chapter 4 (complete) — Midterm Exam
   Y: Chapter 5
6. Intergers
   P: Chapter 6 (complete)
   M: pp. 18 - 19
   Y: pp. 73 - 91 — Test 4
7. Rationals
   P: Chapter 7 — Test 5
8. Real Numbers
   P: Chapter 8 (omit 8.14a)
9. Probability and Statistics
   MS: Chapter 8 — Final Exam

COURSE OUTLINE

AND

TEACHING GUIDE

T104
General Mathematics
for
Elementary Teachers

July 1968

I. Introduction

This course is primarily for students with one year of study in each of high school algebra and high school geometry. Students with such preparation should be able to take the course with profit and without major difficulty. Students who have not had the equivalent amount of prior study may need to be supplied some background material. A person with a good command of the concepts, proofs, and techniques presented in the course should have the understanding of arithmetic necessary to teach in the elementary school.

Since it is intended that the course be thoroughly mastered, a superficial acquaintance with the concepts and techniques presented here will not be sufficient. A large part of the mathematics of the elementary school is arithmetic; consequently the preparation must assure that a teacher have understanding and skill in arithmetic and confidence in his knowledge of the basic concepts of the subject.

Although familiarity with the natural numbers might seem to make unnecessary any attempt to define these numbers, we have sought to go a little deeper and to relate the fundamental notion of numbers to the correspondence between sets. The real numbers are introduced by successive extensions of the set of natural numbers. In each extension, the associative, commutative, and distributive properties are required to hold. The rules for performing operations are obtained by use of these properties and definitions.

The use of simple equations and inequalities should be a prominent feature of the course, and the student should have extensive practice not merely in solving equations and inequalities, but also in formulating them to solve word problems.

It should be understood that this course guide is merely a brief sketch of the content of the topics that should be covered in the order we believe that they should be studied. The brevity of the sketch creates an impression of logical austerity. It is, however, an essential part of the task of the teacher to avoid such austerity by filling in an intuitive background and furnishing illustrations at every opportunity. A bare sequence of definitions, theorems, and proofs is unacceptable for two reasons. In the first place, it would be pedagogically quite hopeless at the level, and for the audience, that we have in mind. In the second place, the prospective elementary teacher needs to become aware of ways to bridge the gap between mathematical ideas as they appear in formal theories and the various intuitive forms in which these same ideas may be introduced to young children.

There is an instructor's manual for Theory of Arithmetic which may help the instructor to decide what material to emphasize. Sample quizzes and answers to problems are included in this manual.

II. Course Outline

1. History and Foundations

   A. Content
   This introductory section includes a brief summary of the history of numerals and systems of numeration. Consider the Egyptian hieroglyphic, Roman, Ionic Greek, Chinese-Japanese, Mayan, and Babylonian systems. It is not necessary for the students to memorize the various symbols, since the emphasis should be on recognizing the characteristics of each system. Discuss the counting board and the abacus as examples of early computing instruments. Examine finger multiplication, patterns of numbers, and geometric patterns. Introduce the tally system as the first system of representing numbers. There is no need to assign exercises in this section. This section will not be tested.

   B. References
   P: 1.1-1.4, 1.5e, 1.5f.
   M: 1.1
   E: selected topics
   S: selected topics

   C. Recommended Time: 2 lectures

2. Logic

   A. Content
   Discuss the difference between simple statements and compound statements. Discuss the meanings of the connectives, and construct simple truth tables. Introduce the idea of implication. Show how a statement, its inverse, its converse, and its contrapositive differ. Briefly discuss quantifiers with emphasis on how to negate quantified statements. Discuss different kinds of everyday logical arguments. Then discuss the nature of proof and various common proof procedures.

   B. References
   M: 9.1-9.4
   Y: 282-290 (required reading)
   Y: 291-301

   C. Recommended Time: 5 lectures

3. Numeration Systems

   A. Content
      Discuss the Hindu-Arabic system, exponents, decimal system, standard notation, and scientific notation. Introduce systems with bases other than ten. Show some examples of computation in other bases, but do not give students too much of this to do on their own. Aim for understanding rather than computational skill. Show how decimal fractions may be written in other bases. Some students may be interested in looking at negative number bases on their own. Such material may be found in the Mathematics Teacher.

   B. References
      P: 1.5a, 1.5b, 1.5c, 1.5d, 1.6
      P: Chapter 5 (complete)
      M: Chapter 2 (required reading)
      Y: pp. 41-49, pp. 234-239
      T: selected topics

   C. Recommended Time: 5 lectures

      Note: Test 1 follows this section.

4. Set Theory and Relations

   A. Content
      Do not assign an excessive number of exercises in this section. Discuss sets, subsets, intersection, union, universal set, empty set, complement, and cartesian product. Discuss relations, one-to-one correspondence, and cardinal numbers. Introduce the concept of a function and show how to graph relations and functions.

   B. References
      P: Chapter 2 (complete)
      M: Chapter 4
      P: Chapter 3 (complete)

   C. Recommended Time: 8 lectures

      Note: Test 2 follows this section.

5. Whole Numbers

A. Content
In this section we show the relation between the cardinal number of a set and the corresponding whole number. Discuss binary relations and the various properties of the whole numbers. Define the system of whole numbers. Show examples of the addition and multiplication algorithms, but do not require the students to do more than one of each. Discuss the order relations for the whole numbers, leading into the notion of an upper bound on a set. Define LUB and GLB in the appropriate manner. Show students how to graph the examples in section 4.16b on the number line. If the students want to look at shortcuts in computation, read Yearbook.

B. References
P: Chapter 4 (complete)
Y: Chapter 5

C. Recommended Time: 10 lectures

Note: Midterm Examination follows this section.

6. Integers

A. Content
Introduce the set of integers as an extension of the set of whole numbers. Define the system of integers. Show distinction between a prime number and a composite number. Prove that the number of primes is infinite. Discuss prime factorization and state the Fundamental Theorem of Arithmetic. Give an example of a system without unique factorization, such as the set of even positive integers. State and give examples of the division algorithm and define GCD and LCM. Discuss order relations among the integers. Distance on the line is given by a discussion of absolute value. Use clock arithmetic to introduce congruence. Another example of a system without unique factorization can now be shown. The set of positive integers congruent to 1 modulo 3 does not have unique factorization. Consider some unsolved problems of number theory.

B. References
P: Chapter 6 (complete)
M: pp. 18-19
Y: pp. 73-91 (optional)

C. Recommended Time: 12 lectures

Note: Test 4 follows this section.

7. Rationals

A. Content
Define a rational number as an equivalence class of ordered pairs of integers. Define the system of rational numbers. Discuss the various interpretations of rational numbers. Discuss the order relations in the rationals and derive the various properties of order. Discuss the property of denseness and plot solution sets of inequalities. Introduce the irrational numbers by looking for a number whose square is two. Prove that the square roots of 2, 3, and 5 are not rational. Ask students to try to prove that the square root of 4 is irrational.

B. References
P: Chapter 7

C. Recommended Time: 12 lectures

Note: Test 5 follows this section.

8. Real Numbers

A. Content
Discuss the real number line. Consider decimal approximations of rational and irrational numbers. Define the system of real numbers. Discuss square roots, omitting the square root algorithm in section 8.14a, and consider Newton's method. While talking about the complex numbers, keep the discussion as simple as possible.

B. References
P: Chapter 8 (omit section 8.14a)

C. Recommended Time: 7 lectures

9. Probability and Statistics

A. Content
Define probability and give examples that can be solved by counting methods. Do not introduce permutations and combinations. Show the difference between mean, median, and mode. Give examples of different distributions to show this. Show how the standard deviation can be found. Discuss correlation intuitively by means of graphs of points. Show the standard normal curve and its characteristics. Give examples of how statistics can "lie".

B. Reference
MS: Chapter 8

C. Recommended Time: 5 lectures

Note: Final Examination follows this section

10. Optional Topics

These topics may be taught if time allows. They are not designed to be taught at the end of the course; rather they should be inserted at appropriate points in the course. The decision as to which of these topics should be taught and when they should be taught will be left to the instructor.

A. Four Color Problem
Meserve & Sobel, Introduction to Mathematics, chapter 1, section 4, p. 19.
Meserve & Sobel, Mathematics for Secondary Teachers, p. 348.
Mathematics Teacher, May 1967, pp. 516-519

B. Negative Number Bases
Mathematics Teacher, November 1967, pp. 723-726.

C. Fermat's Last Theorem and Goldbach's Conjecture
Meserve & Sobel, Introduction to Mathematics, chapter 1, section 4, p. 18-19.

D. History of Mathematics
Film: Donald Duck in Mathemagic Land, available from I.U. Audio-Visual Department.
Mathematics Teacher, March 1967, pp. 264-278.

E. Geometry
See book by Abbott, Flatland.

F. Secret Codes
Peck, Secret Codes, Remainder Arithematic and Matrices, NCTM, 1961.

G. How To Lie With Statistics
See book of same name by Huff.

H. Matrices (2 x 2)
Introduction to Matrix Algebra, SMSG.

I. Applications of Probability To Games
Mathematics Teacher, March 1967, pp. 210-214.

J. Friendly Numbers
   Mathematics Teacher, February 1967, pp. 157-160.

K. Mathematics and Music
   Mathematics Teacher, March 1968, pp. 268-271.

L. Digital Problems
   Mathematics Teacher, February 1968, pp. 181-189.

M. Magic Squares
   27th NCTM Yearbook, pp. 207-220
   Mathematics Teacher, January 1968, p. 18.

Grading Standards for T104

To facilitate uniform grading in T104 it is suggested that a "C" grade should require mastery of the following objectives.

1. a) Be able to write any given real number in expanded notation.
   b) Convert numbers to scientific notation and the reverse.
   c) Work problems with exponents, i.e.

$$3^2 \cdot 3^4 = \underline{\ ?\ } \qquad (3^2)^2 \overset{?}{=} 3^4$$

   d) Convert integral numerals in other bases into numerals in base 10.
   e) Be able to perform simple addition, subtraction, multiplication, and division in the various bases.

2. a) Be able to write definition of union, intersection of sets, complement of a set, relation, function, and properties of equivalence relation.
   b) Determine union, intersection of certain sets and the complement of a set.
   c) Given information about a particular set, list the elements or describe the set.
   d) Determine whether a given relation satisfies properties of an equivalence relation and maybe determine equivalence classes.
   e) Be able to tell if two sets are in 1-1 correspondence.

3. a) Be able to write definitions of closure, commutativity, and associativity.
   b) Compute cardinal number of finite sets.
   c) Plot solution sets on number line.
   d) Write definition of identity and inverse properties.
   e) Determine whether a given number system satisfies the five above properties.
   f) Fill in the reasons in the steps in the addition, subtraction, multiplication and division algorithms.
   g) Write definition of addition, subtraction, multiplication and division for whole numbers.

4. a) Do computation in integers. Therefore, must know laws concerning addition and multiplication.
   b) State Fundamental Theorem of Arithmetic.
   c) Find GCD and LCM.

5. a) Do computation in rational numbers using equivalence class definition.
   b) Order any given set of rational numbers.
   c) Determine if a given number is irrational or not.
   d) Work problems in percent, i.e., 20% of 10 = ___ , 36 is what percent of 40?

6. a) Know that there exists a number for every point on the line.
   b) Convert rational numbers to decimal expanded form (1/3, $.3\overline{3}$) and back the other way.

7. a) Give definition of probability, mean, median, and mode.
   b) Compute probahility of any event that can be determined by simple counting procedures.

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E Eves, Howard, An Introduction to the History of Mathematics. Holt, Rinehart, and Winston, New York, 1953.

M Meserve, B. E. and Sobel, M. A., Introduction to Mathematics. Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1964.

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P Peterson, J. A. and Hashisaki, J., Theory of Arithmetic (second edition). John Wiley and Sons, Inc., New York, 1967.

R Rees, P. K., Principles of Mathematics. Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1959.

S Smeltzer, D., Man and Number. Collier Books, New York, 1962.

T Terry, G. S., Duodecimal Arithmetic. Longman, Green and Co., New York, 1938.

Y National Council of Teachers of Mathematics, Enrichment Mathematics for the Grades, Twenty-Seventh Yearbook, National Council of Teachers of Mathematics, Washington, D. C., 1963.

INTRODUCTION TO LOGIC

Prepared for T104
by
Ralph Seifert, Jr.

INTRODUCTION

Mathematicians spend most of their time discovering new mathematical facts and creating arguments (or proofs) to show why these facts should be accepted. We hope that in this course you will discover many mathematical facts which are new to you. In this mimeographed material we shall explore the activity of proof construction by trying to answer the question, "What is a convincing argument?"

Arguments use language, of course, and some arguments may use sentences which are structurally quite complex, such as: "If both the cost of soap and the cost of detergent go up, then the laundries will either raise rates or stop free deliveries; but if the cost of soap stays the same, or if the cost of detergent stays the same, then the laundries can be run as before, unless....." Whether or not an argument is convincing depends on whether or not certain sentences used in the argument are true. It may be hard to tell whether or not a long, complex sentence is true. In the first part of this unit we will learn how to determine the truth or falsity of a long, complex sentence by looking at shorter, simpler sentences. As a means toward this end we will analyze certain logically important English words like "and", "or", and "not", with a view toward determining their exact meanings in everyday speech.

Next, we will discuss different kinds of arguments and get some practice in telling good ones from bad ones. Finally, we will look at some special arguments used in mathematics (and show that they are convincing).

In this study of everyday arguments we will use mathematical methods. Therefore the very first section of this material explains how mathematics can be used to study real everyday situations. (You may also have use for this information when you study problem solving.)

HOW MATHEMATICS IS USED

Mathematics, as practiced today, can be defined as the study of abstract structures. Despite this, our main use of mathematics is in the study of real objects; as one grade-school student put it, "Because of numbers, we can figure out what happens if we have ten apples and do something." Exactly how do we "figure out what happens"?

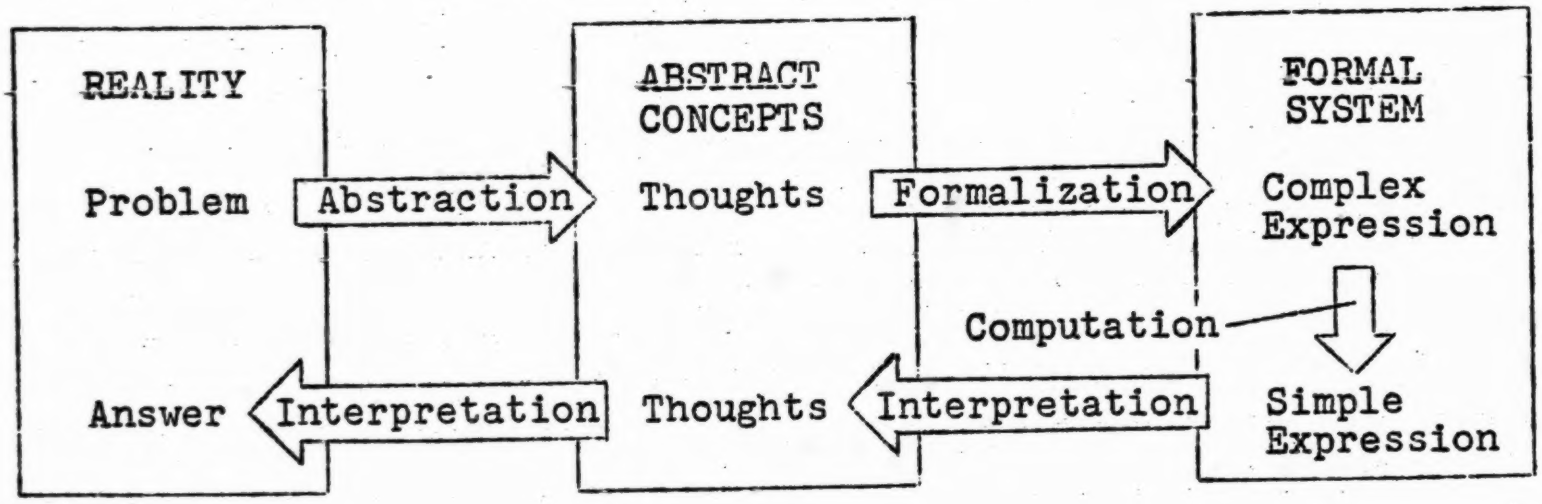
Suppose we have ten apples and do something; say we decide to throw away the three smallest apples and cut each of the remaining ones into halves. I ask how many pieces of apple we will then have. Instantly you think, "ten minus three, multiplied by two." You have taken the first step in solving the problem, namely that of abstraction. That is, you have correctly noticed that the solution to the problem does not depend on the fact that the objects involved are particular apples, or on the fact that the discarded apples are to be the smallest ones (the answer would be the same for any other ten apples, or for peanuts, or kumquats, or sponges); the answer only depends on the abstract concepts of (in this case) ten-ness, three-ness, throwing-away, and cutting-in-two. At the same time, you have given names to these concepts; you might even have written down symbols to help you keep things straight, e.g.

$$(10 - 3) \times 2 = ?$$

This is the step of formalization, or the representation of abstract ideas by means of symbols or words.

The next step is computation, or the reduction of complex formal expressions to simpler ones. You know that "three from ten is seven" and that "seven times two is fourteen". Notice that if you wish, you can perform this step without reference to the abstract concepts you were thinking about a moment ago; for you know that in this system the symbol "(10 - 3)" can simply be replaced by the symbol "7", and that "7 x 2" can be replaced by "14".

The final step is interpretation. You interpret the formal symbol "14" as denoting the abstract concept of fourteen-ness, and then you interpret this concept in terms of the original physical situation and announce that the answer is "fourteen pieces of apple". Probably the entire four-step process took only a few seconds, and you were not aware of the individual steps. This is because the problem was so simple.


REALITY
Problem
Abstraction
ABSTRACT CONCEPTS
Thoughts
Formalization
FORMAL SYSTEM
Complex Expression
Computation
Simple Expression
Interpretation
Thoughts
Interpretation
Answer

EXERCISE. Think of how the four steps mentioned above are carried out when solving more difficult problems, like: (1) Filling out an income-tax form. (2) Designing a suspension bridge.

THE LOGIC OF EVERYDAY LANGUAGE

Let us apply the ideas of the preceding section to a study of the following problem: given an English sentence, we ask whether or not it is true. You might not have thought this was a mathematical problem, since it does not involve numbers. But languages have structure, so they are to that extent capable of mathematical analysis.

Statements. English sentences fall into two classes, namely the class of sentences that are neither true nor false, and the class of sentences which are either true or false. The first class contains such sentences as:

Come back tomorrow.
Where are you?
Help!
The next sentence is false.
The preceding sentence is true.

The second class contains such sentences as:

Gold is a metal.
George likes cake.
$2 + 6 = 8$.
$2 + 6 = 9$.
If I fall in the lake, I will get wet.

Everyday logical reasoning uses only sentences from the second class. Thus we define a statement to be a sentence which is either true or false, and we confine the rest of our discussion to the class of statements. A statement which is true will be said to have truth-value T; false statements have truth-value F.

A statement may contain simpler statements within itself; for example, the statement "John is downstairs and Fred is outside" contains the simpler statements "John is downstairs" and "Fred is outside". In such cases the truth-value of the entire statement clearly depends both upon the truth-values of the simpler statements and upon the nature of the word or words used to connect them (in this case, "and"). Words like "and", which have no substantive meaning themselves but are used only to join two statements together, are called connectives. For example, the words "and", "but", "either...or", "neither...nor", and "if... then" can be used as connectives. A statement which is, or can be rewritten as, a combination of two or more statements by using connectives is called a compound statement; statements which are not compound are called simple. The simple statements which make up a given compound statement will be called its atoms.

The truth-value of some statements just cannot be decided;

for instance, consider "Exactly 104 Sioux Indians were born in 1487." Therefore the problem we first mentioned (that of determining the truth-value of any given statement) cannot be solved by mathematical methods. Suppose, however, that our given statement is compound, and that the truth-value of each of its atoms is known. Then the truth-value of the entire statement depends only on what the connectives are, and in what order they occur. This problem begins to look like a mathematical one; that is, we can conceive of a mechanical computation which will tell us the truth-value of the compound statement, given the truth-value of each of its atoms. To decide how to do this computation, we must consider each connective individually and decide how it is used (i.e. what its use means) in everyday speech.

The connective "and". We begin our analysis of connectives by deciding how to find the truth-value of a statement of the form "A and B", where "A" and "B" are statements whose truth-values are known. Suppose you are sitting in an office and your friend Sam suddenly points to two of the desks in the office and shouts, "There is a flamingo sitting on this desk, and that other desk is on fire!" You carefully examine both desks and see that there is indeed a flamingo sitting on the first desk, and that the second desk is actually on fire. You would then certainly agree that Sam's statement was true, and you would call the firemen to come put out the fire, and tell the zoo to come catch the flamingo. On the other hand, suppose you saw no flamingoes sitting on the first desk, and the second desk was not on fire. You would then say that Sam's statement was false (or even "totally false"), and you would call neither the firemen nor the zoo. (You might call a psychiatrist for Sam.)

What would you say (regarding Sam's statement) if there was a flamingo sitting on the first desk, but the second desk was not on fire? What would you say if the second desk was on fire, but the first desk bore no flamingoes? In each of these situations you would probably say that Sam's statement was at best "only half true", "only partly true", or "partly false". That is, although you would call the zoo (in the first case) or the firemen (in the second case), you would say that Sam's statement was not entirely true; you might say that it was "technically false" or "false, in the broadest sense of the word". In logic, "false" means "not entirely true"; we do not distinguish between the everyday notions of "partly false" and "totally false". So mathematicians would say that Sam's statement was just false.

The decisions we have reached can be conveniently summarized in a table, such as the following:

If the truth-value of "There is a flamingo sitting on the first desk" is	and the truth-value of "The second desk is on fire" is	then the truth-value of "There is a flamingo sitting on the first desk, and the second desk is on fire" is
T	T	T
T	F	F
F	T	F
F	F	F

A table of this sort, which gives the truth-value of some statement for each possible combination of truth-values of its atoms, is called a truth table for that statement.

A completely similar discussion applies to any other statement involving "and"; that is,

If "X" is any statement whose truth-value is	and "Y" is any statement whose truth-value is	then the truth-value of the statement "X and Y" is
T	T	T
T	F	F
F	T	F
F	F	F

In words, the statement "X and Y" is true when both "X" and "Y" are true, and "X and Y" is *false* in all other cases.

Formulas. The expression "X and Y" is an example of a *formula*. In general, a (logical) formula is an expression, containing *variables* (like "X", "Y", etc.) and connectives, *which would become a statement if statements were put in where the variables are*. The distinction between statements and formulas is like the distinction between numbers (2, 3, etc.) and variable terms ($x+y$, $4z$, etc.). Formulas can be thought of as abbreviations for statements, in the sense that we use single letters (variables) to stand for the atoms of the statement. When computing the truth-value of a compound statement from the truth-values of its atoms, we ignore the actual *content* of the ~~atoms and consider only~~ *what the* ~~*connectives are*~~ and *how they connect* the atoms (i.e. in what order). An appropriate formula can give all *this* information, so we can work with formulas instead of statements. Since each formula abbreviates thousands of different statements, we can thus analyze all these thousands of statements at once. This is the advantage of using formulas.

The *truth table* of a formula is a table like the last one given above, in which we can look up the truth-value of any statement abbreviated by the formula, given the truth-values of the statements represented by the variables in the formula.

The connective "or". We proceed to analyze other connectives; next we look at "either...or". Suppose Jack says, "Either smoking is prohibited or drinking is prohibited." What would you say (regarding the truth-value of this statement) if it turned out that smoking was not prohibited, but drinking was? If smoking was prohibited, but drinking was not? If neither one was prohibited? If both were prohibited? Make a truth table for the formula "X or Y" which summarizes your decisions.

In the above problem you may have had trouble deciding what to say if both smoking and drinking were prohibited. Just what Jack meant to say about this possibility is not entirely clear. This is because the word "or" is used in English in two different ways. Jack might have meant "Either smoking is prohibited or drinking is prohibited (or possibly both); this usage of "or" is called the inclusive usage. On the other hand, he might have meant, "Either smoking is prohibited or drinking is prohibited (but not both)"; this usage is called the exclusive usage. So we have

X	Y	X or Y (inclusive)	X or Y (exclusive)
T	T	T	F
T	F	T	T
F	T	T	T
F	F	F	F

In everyday speech we often have to determine from the context which way the word "or" is being used. If a man says, "I have lots of things to buy, so I will go downtown today or tomorrow", you would probably agree that he might actually go on both days, or at least that he had not said he would not. So this is an inclusive usage. On the other hand, if a man says, "By this time tomorrow I will either be rich or bankrupt", he is obviously using "or" in the exclusive sense; for wealth and bankruptcy cannot occur at the same time.

When a mathematician says "or", he always means the inclusive usage (unless he says otherwise). This agreement is made under the general principle that in mathematics a given word should have only one meaning. Thus from now on "or" means "inclusive or", regardless of the context; and we write the table

X	Y	X or Y
T	T	T
T	F	T
F	T	T
F	F	F

Negation. The word "not" is not used to join two statements together, but it is like the other connectives we have studied in that the truth-value of the statement "not A" is completely determined by the truth-value of "A". In fact we have the truth table

X	not X
T	F
F	T

Therefore we shall call "not" a connective also.

Combinations of connectives. When using more than one connective, it is necessary to indicate precisely which statements are connected by which connectives. You recall that in arithmetic we cannot write just "3 x 4 + 2" without being ambiguous; we must either use parentheses or else come to some agreement about the order in which x and + are to be considered (usually we agree to consider x first). The problems are similar here; what does "A and B or C" mean? In writing English statements we avoid this sort of ambiguity by using punctuation and various literary devices. In writing logical formulas, we use parentheses. As in algebra, when interpreting an expression involving parentheses we work outwards from the innermost pair(s) of parentheses.

Statement: Either both Tom and Jim will go, or Bill will.
Formula: (X and Y) or Z

Statement: Tom is going, and either Jim or Bill is going.
Formula: X and (Y or Z)

Statement: Jack and Fred are not both going.
Formula: not (X and Y)

Statement: Jack is not going, but Fred is.
Formula: (not X) and Y

The connective "if...then". Suppose that at 11:00 a.m. Leo says, "If you wind your watch now, then it will still be running at noon." If you immediately wind your watch and it is, in fact, still running at noon, you would certainly say that Leo's statement was *true*. On the other hand, if you immediately wind your watch but find at noon that it has stopped, you would certainly say that Leo's statement was *false*. The same sort of discussion applies to other "if...then" statements; so on the basis of everyday experience, we all agree that the following truth table (so far as it goes) is accurate:

X	Y	if X, then Y
T	T	T
T	F	F

What shall we put in this truth table when "X" has truth-value F? To return to the above example, what would you say about Leo's statement if it happened that you did *not* wind your watch at 11:00?

Suppose first that the watch was still running at noon. You might say that since the watch is running anyway, it certainly would still be running even if you *had* wound it; so Leo's statement was *true*. But this argument is not convincing. For one thing, if you had wound the watch you might have wound it too tightly, making it stop; or it might have stopped for some other reason, not connected with your winding it. For example, if you had wound the watch at 11:00, you might have been *run over by a herd of elephants* at 11:30, causing the watch to stop. You really *cannot know* what would have happened if you had wound the watch at 11:00, simply because you *did not do so*. However, this does not prove that Leo's statement was *false*, for it is still possible that your watch *would* have been running at noon if you *had* wound it.

Similarly, if your watch is in fact *not* running at noon,

[illegible]

you still cannot know what would have happened if you had wound it at 11:00. So we have not yet settled the question of whether Leo's statement was true or false if you did not wind the watch at 11:00. By now you may be thinking that it does not matter whether we call Leo's statement true or false, since it is irrelevant(or has no useful content) in this situation (in which the watch was not wound). For computational purposes, however, we want "if X, then Y" to *have* a truth-value when "X" has truth-value F; so we must choose a *reasonable* one. The following remarks attempt to justify the choice which mathematicians have agreed to make.

Consider the following pairs of statements:

"If I don't pass T104, I will have to leave school."
"Either I pass T104 or I will have to leave school."

"If I am not here at 6:00, then I will be at home."
"At 6:00 I will be either here or at home."

"If I am not crazy, then *you* must be."
"One of us is crazy."

In each pair you should agree that the first statement and the second statement have the same *import* or *content*, or that they *convey the same information*; that is, if you were in a position to use one of them, it would not matter much which one you used. We could also say that the two statements in each pair *have the same meaning*. As a consequence of this, the two statements in each pair have the *same truth-value*. Now notice that the first statement in each pair is of the form "if (not Z), then Y"; and the second statement is of (or can be expressed in) the form "Z or Y", where of course each of "Z" and "Y" stands for the same statement in both formulas. (For example, in the third pair "Z" is "I am crazy" and "Y" is "You are crazy".) Thus we have some support for the assertion that *if* "Z" *and* "Y" *are statements*, *then* "if (not Z), then Y" *means the same thing as*

"Z or Y". In particular, letting "Z" be the statement "not X", we see that "if(not(not X)), then Y" means the same thing as "(not X) or Y". But to say "not (not X)" is to say "X"; so "if X, then Y" means the same thing as "(not X) or Y".

We have already studied "or" and "not", so we can make the following computation:

If "X" and "Y" are both true, then "not X" is false and "Y" is true; so "(not X) or Y" is true.

If "X" is false and "Y" is true, then "not X" is true and "Y" is true; so "(not X) or Y" is true.

(You can finish this computation.)

Since "if X, then Y" means the same thing as "(not X) or Y", we can use this computation to write the following truth table:

X	Y	if X, then Y
T	T	T
T	F	F
F	T	T
F	F	T

Notice that the first two rows of this table agree with the partial table we put down before.

When writing formulas, we often write "X → Y" as an abbreviation for "if X, then Y". Thus we have the table

X	Y	X → Y
T	T	T
T	F	F
F	T	T
F	F	T

In words, "X → Y" is false just when "X" is true and "Y" is false. Thus (and this is very important) the truth of the statement "if A, then B" does not, by itself, imply any sort of *connection* (logical or causal) between the *contents* of the statements "A" and "B". For example, the statement "If cows give milk, then birds fly" is *true*, since "Cows give milk" and "Birds fly" are both true. But we cannot *deduce* "Birds fly" from "Cows give milk"; nor can we say that the flight of birds is *caused* by the giving of milk by cows.

EXERCISES. 1. Find the truth-values of the following statements: (a) $2 + 4 = 5$ and $2 + 5 = 7$; or else $3 + 1 = 4$. (b) If $3 < 2$, then both $3 < 4$ and $3 < 1$. (c) Either $4 < 6$ or $6 < 4$; and furthermore, $10 + 5 = 12$.

2. Find a number x such that the following statement is true: $x < 2$ or $x > 5$; and also $x > 0$ and $x < 1$.

3. Assuming that Joe, Sam, and Jane *are* going to the party and that Mary and Fred are *not* going to the party, which of the following statements are true? (a) Joe is going to the party and Mary is not going to the party. (b) If Sam is not going to the party, then neither Fred nor Jane will go. (c) If either Mary or Jane goes to the party, then Sam will not go to the party.

4. How did you (or how could you) use abstraction, formalization, and interpretation in obtaining your answers to Ex. 3?

Tables for combinations of connectives. A good way to compute the truth table of a formula containing several connectives, such as "X and Y, or not X", is to make a table with several columns (see the table below). In the first two (or more) columns we place all the possible combinations of truth-values for "X" and "Y" (and any other variables which appear). For each such combination (reading horizontally) we then compute, one step at a time, the truth-value of the entire formula (in this case "X and Y, or not X"), writing down each step in the appropriate column. In this particular case we proceed as follows in each row: (a) in the third column, we put the result of applying "and" to the given values for "X" and "Y"; (b) in the fourth column, we put the result of applying "not" to the given value of "X"; and finally (c) to fill the fifth column, we apply "or" to the previously computed values of "X and Y" and "not X". To better understand this process, compare it with the process of computing (x × y) + (-x) for various numbers x,y.

X	Y	X and Y	not X	(X and Y) or (not X)
T	T	T	F	T
T	F	F	F	F
F	T	F	T	T
F	F	F	T	T

Thus, for example, we can use the table to see that if "A" and "B" are both false statements, then "A and B, or not A" is true statement.

Two more examples:

1. Truth table for "if X, then not (X and Y)".

X	Y	X and Y	not (X and Y)	X → not (X and Y)
T	T	T	F	F
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

2. Truth table for "(X and Y) or (Z and Y)".

X	Y	Z	X and Y	Z and Y	(X and Y) or (Z and Y)
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	F	F
T	F	F	F	F	F
F	T	T	F	T	T
F	T	F	F	F	F
F	F	T	F	F	F
F	F	F	F	F	F

In the preceding section we remarked that if two statements had the same meaning, then the (last columns of) the truth tables of the corresponding formulas would be the same. This remark works both ways. For example, the following shows that "X or Y" has the same truth table as "not ((not X) and (not Y))".

X	Y	not X	not Y	(not X) & (not Y)	not((not X) & (not Y))	X or Y
T	T	F	F	F	T	T
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	T	F	F

In other words, if "A" and "B" are statements, then the statements "A or B" and "not ((not A) and (not B))" will have the same truth-value, regardless of the actual truth-values of "A" and of "B". Thus we can say that "A or B" and "not ((not A) and (not B))" mean the same thing, or that "or" can be expressed in terms of "not" and "and". Two formulas containing the same variables are said to be equivalent if (the last columns of) their truth tables are the same.

EXERCISES. 1. Show that "not (A and B)" means the same thing as "if A, then not B", for any statements "A" and "B".

2. Decide what the truth table of the connective "neither... nor" should be, and then show that "neither A nor B" means the same thing as "(not A) and (not B)".

3. Show that "A and B" means the same thing as "B and A", and that "A and (B or C)" means the same thing as "(A and B) or (A and C)". What algebraic laws do these remind you of?

4. Show that "and" can be expressed in terms of "not" and "or"; i.e. show that "X and Y" is equivalent to some formula whose only connectives are "not" and "or" (possibly used more than once).

5. Is "X and (if X, then Y)" equivalent to any simpler (i.e. shorter) formula?

6. Find a formula (the last column of) whose truth table contains only T's. What can you say about a statement that is represented by such a formula?

The connective "if and only if". Statements of the form "A if and only if B" are used frequently in mathematics; so frequently, in fact, that mathematicians have taken to abbreviating such statements by "A *iff* B". However, the connective "if and only if" is seldom used in everyday speech; so we shall discuss its meaning. Now "A, if B" means "if B, then A"; and "A only if B" means "A only holds if B also holds", or "whenever A holds, then also B holds", or just simply "if A, then B". So "A if and only if B" means "(if A, then B) and (if B, then A)". The truth table for the corresponding formula is

X	Y	X → Y	Y → X	(X → Y) and (Y → X)
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Hence the truth table for "iff" is

X	Y	X iff Y
T	T	T
T	F	F
F	T	F
F	F	T

Thus the statement "A iff B" can be taken to mean that "A" and "B" *have the same truth-value*, or that the situations described by "A" and by "B" *always occur together*; that is, if either occurs, then the other also occurs. Example: "A polygon has three sides iff it has three corners". Again, however, the truth of "A iff B" does not carry with it any claim of a logical connection between "A" and "B". For example, "Birds have legs iff ice floats on water" is a true statement.

If "A" and "B" are statements describing conditions and "if A, then B" happens to be *true*, then we say that the condition described by "A" is a *sufficient condition* for the presence of the condition described by "B", and that "B" is a *necessary condition* for "A". That is, "if A, then B" says that for "B" to be true, it is sufficient for "A" to be true; it also says that if "A" is true, then it is necessary that "B" be true. If "A iff B" is true, then each of "A" and "B" is called a *necessary and sufficient condition* for the other. These terms were more widely used ~~back in the days when long words were preferred to short ones.~~

Inverse, converse, and contrapositive. Suppose "C" is the statement "if A, then B". Then the *converse* of "C" is the statement "if B, then A"; the *inverse* of "C" is "if (not A), then (not B)"; and the *contrapositive* of "C" is "if (not B), then (not A)".

EXERCISES. 1. Make up two English "if...then" statements and write their converses, inverses, and contrapositives.

2. Show that an "if...then" statement means the same thing as its contrapositive.

3. Show that the converse and the inverse of an "if...then" statement mean the same thing.

4. Show that an "if...then" statement does *not* always mean the same thing as its converse.

Quantifiers. We have seen that statements containing connectives can be broken down into simpler statements, and that the truth-value of the original statement can be computed from the truth-values of the simpler statements. At the same time, we have analyzed the "meanings" of the various connectives. There are many statements which are, in a sense, made up of simpler statements, but which might not involve explicit occurrences of connectives. These are the statements involving such words as "every", "each", "any", "some", and so on, used to express one of the two ideas

"Every P is a Q" or
"Some P is a Q",

where P and Q are names or descriptions of objects. The words "every" and "some", when used in this way, are called _quantifiers_.

The meaning of each quantifier is clear. "Every P is a Q" means "p_1 is a Q, and p_2 is a Q, and p_3 is a Q, and ...", and "Some P is a Q" means "p_1 is a Q, _or_ p_2 is a Q, _or_ ...", where $p_1, p_2, p_3, \ldots$ is a list of all the P's. This list could be quite long, or even infinite; so we do not attempt to write general truth tables for "every" or "some".

The important thing to know about quantifiers is that (1) "_Not_ every P is a Q" means the same as "Some P is _not_ a Q"; and (2) "_Not_ (some P is a Q)" means "Every P is a non-Q", or "No P is a Q". (Convince yourself of this.) Therefore, one may (for example) show that "Every P is a Q" is _false_ by showing that some P is not a Q.

EXAMPLE. Suppose the algebraic equation $x + yz = (x+y)(x+z)$ is presented as an alleged algebraic rule, and we are asked to prove or disprove it. The fact that the equation is being proposed as a _rule_ means that the proposer is really asserting the statement "For all numbers x, y, and z, we have $x + yz = (x+y)(x+z)$." Some ingenuity may be required to transform given English statements into quantifier statements with the same meanings. In this case we can rewrite the "rule" as "Every three numbers x, y, and z are numbers for which $x + yz = (x+y)(x+z)$." A little experimentation with various numbers shows that this statement is _false_; for example, $1 + (1 \times 1) \neq (1+1) \times (1+1)$. So we can assert that "Some three numbers (namely $x = 1$, $y = 1$, and $z = 1$) are _not_ numbers for which $x + yz = (x+y)(x+z)$." By the above remarks, this is the same as saying that "Not every three numbers x, y, and z are such that

$x + yz = (x+y)(x+z)$." So we have *disproved* the rule. However, we may not now just write $x + yz \neq (x+y)(x+z)$, since this could be taken to mean that *every* three numbers x, y, and z are such that $x + yz \neq (x+y)(x+z)$, which is not what we want to say. (In fact $1 + (0 \times 0) = (1 + 0) \times (1 + 0)$ and $\frac{1}{2} + (\frac{1}{3} \times \frac{1}{6}) = (\frac{1}{2} + \frac{1}{3}) \times (\frac{1}{2} + \frac{1}{6})$.)

EXERCISES. Express the negations of each of the following statements in good English. Avoid ambiguity.

1. A rolling stone gathers no moss.
2. Someone here has a book.
3. Someone here does not have a book.
4. The sun will shine every day next week.
5. All rulers in the U.S. are marked off in inches.

THE NATURE OF ARGUMENT

An argument can be defined as a *discourse whose purpose is to convince people that some given statement is true*. The statement which the argument tries to prove is called the *conclusion* of the argument, and the statements which claim to support the conclusion are called *premises*. An argument is *convincing* if people who believe the premises to be true agree, after reading the argument, that its conclusion is also true. There are two major types of argument, called *inductive* and *deductive*.

Inductive arguments. Suppose a zoologist wants to prove the statement "Koala bears eat only eucalyptus leaves". He gets a government grant, goes to Australia, and observes a total of 5684 different koala bears, *none* of whom he sees eating anything but eucalyptus leaves. He then says, "Each of the 5684 koala bears I saw ate only eucalyptus leaves while I was watching; therefore all koala bears eat only eucalyptus leaves." Assuming that we believe his premises to be true, just how convincing this argument is depends on how many koala bears there are in the world, and on how thoroughly this zoologist observed the ones he saw. If we knew that there were only 6000 koala bears in the world, and if the zoologist had observed each of his samples for an entire year, then we would tend to accept his argument. If we knew that 1,000,000 koala bears existed, or if the zoologist had only watched each animal for five minutes, we would probably not be convinced. But in no case are we completely convinced that all koala bears limit their diets to eucalyptus leaves. We are willing to believe at most that it is a good bet that the next time you see a koala bear eating, he will be eating eucalyptus leaves. An argument of this kind, which attempts to prove some general statement by citing a long (but not exhaustive) list of specific cases, is called an inductive argument. As the above example indicates, we can never be 100% sure that the conclusion of an inductive argument is true, even if we are 100% sure that each of the premises is true; rather, the conclusion of such an argument must be interpreted as a statement that something is a "good bet". (Of course scientists do interpret their conclusions in this way, even when they are expressed as universal assertions.) Even though inductive arguments do not lead to absolute truths, they are the best arguments we can get for many statements about the real world.

Deductive arguments. A deductive argument is one which takes all its force from the linguistic structure of its premises, rather than from their content. Thus the power of a deductive argument to convince is the same, regardless of the meanings given to its individual words. It is possible to write deductive arguments which are absolutely convincing, in the sense that mature, rational people will agree that if the premises are 100% true, then the conclusion must be 100% true. Such arguments are called valid. Non-convincing deductive arguments (in which it is possible for the conclusion not to be 100% true even when the premises are 100% true) are called invalid.

There are many kinds of deductive arguments used in everyday life, and we will not try to list all of them here. Most college students can accurately classify a deductive argument as valid or invalid if they can ignore the content of the argument and look only at its linguistic structure. We will give some examples and show how arguments can be tested by mathematical methods.

EXAMPLE. "If I go to the party, I will see Mary. I will go to the party. Therefore, I will see Mary." Let "A" be the statement "I will go to the party" and let "B" be the statement "I will see Mary". Then the premises of this argument are "if A, then B" and "A", and the conclusion is "B". If this is a deductive argument, then its strength lies only in its structure and thus we can establish its validity or invalidity just by looking at the premises and conclusion in this abbreviated form. That is, we ask "If both 'if A, then B' and 'A' are true, must 'B' be true?" By the meaning of "if...then", this is the same as asking if the following statement is true: "If ((if A, then B) and A), then B". Consider the following truth table:

X	Y	X → Y	(X → Y) and X	((X → Y) and X) → Y
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Every entry in the last column is T. This means that for any statements "A" and "B", be they true or false, the statement "if ((if A, then B) and A), then B" is true. Consequently, by the meaning of "if...then", we see that "B" must be true whenever "if A, then B" and "A" are both true. So the quoted argument is completely convincing, and by virtue of its structure alone (since that was all we looked at). Hence the argument is deductive and valid.

It is very important to note that the validity of an argument does not by itself imply that the conclusion of that argument is true. Validity just means that if the premises are true, then the conclusion must be true. If one of the premises is false, then the conclusion need not be true, even though the argument is valid. Thus in the above example, the speaker may be lying when he says he will go to the party; if this is the case, then he might not see Mary. But his argument is still *valid*. Similarly, the fact that some argument is *invalid* does not imply that the conclusion of that argument is *false*.

EXAMPLE. "If I go to town, I will buy something. If I buy something, I will come home with less money. Therefore, if I go downtown, I will come home with less money." The validity of this argument depends on the truth of statements of the form "If ((if A, then B) and (if B, then C)), then (if A, then C)." The truth table

X	Y	Z	X → Y	Y → Z	(X → Y) and (Y → Z)	X → Z	((X → Y) and Y → Z)) → (X → Z)
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

shows that such statements are always true. So the argument is valid.

EXAMPLE. "If Jones were a Communist, he would have voted for this measure. Jones did vote for this measure. Therefore Jones is a Communist." The validity of this argument depends on the truth of statements of the form "If ((if A, then B) and B), then A". The truth table

X	Y	X → Y	(X → Y) and Y	((X → Y) and Y) → X
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

shows that such statements need not be true; that is, it is possible that "if A, then B" and "B" can both be true at the same time that "A" is false (namely in case "A" is false and "B" is true). Therefore this argument is *not* valid. (But again, this does not in itself mean that the conclusion of the argument is false.)

The axiomatic method. As we have seen, the conclusion of a valid deductive argument is accepted as true if it is agreed that its premises are true. If someone doubts that the premises are true, it may be possible to produce another valid deductive argument which has as its conclusion the premises of the preceding one, and has as its own premises statements which the doubter is willing to believe. If these new premises are not accepted by the skeptic, it may be possible to repeat this process again and again until the original conclusion is shown to follow (by a very complicated deductive argument combining several simpler ones) from some set of premises which the skeptic accepts.

Suppose now that we have a very large body of knowledge which we have obtained by direct observation and/or *inductive* arguments. Suppose it also happens that various facts in this body follow from various others by (valid) *deductive* arguments. Finally, suppose we want to present this knowledge to the world and have everyone accept it as true. In such situations it makes sense to try to organize the known facts so that *as few of them as possible* can be verified *only* by observation and experiment, and *the rest follow from these few by valid deductive arguments*. If this is done, then to convince the skeptic that all our results are true, we need only convince him that those few basic ones are true, and that our arguments are valid. This simplifies matters, since verifying a statement by experiment can be difficult, tedious, and/or expensive, while verifying the validity of an argument is easy.

The classic example of this kind of organization is the work of Euclid, a Greek mathematician working in Egypt around 300 B.C. Euclid arranged *all the mathematics which was then known* so that it followed deductively from simple premises like "things equal to the same thing are equal to each other", "a straight line may be drawn through any two points", and so on, which he thought everyone would believe on the basis of everyday experience.

Now when Euclid said "point" or "straight line", he meant to indicate those objects or concepts which everyone ordinarily thinks of when he hears the words "point" or "straight line"; and we might say not just "Euclid's results are true if you accept his premises", but rather "Euclid's results are true if you accept his premises and interpret the words he uses in the way everyone ordinarily does." You may think this idea is too trivial to bring up, since obviously the truth of a statement depends on how we interpret the words in the statement. But suppose we interpret the words "point" and "straight line" to mean "point on the Earth's surface" and "great circle on the Earth's surface". (A great circle is the intersection of the surface with a plane passing through the center of the Earth; the Equator, for example.) Experimentation convinces us that most (though not all) of Euclid's premises are still true under this interpretation of the words. Any of Euclid's results which were proven only from those premises are therefore true under the new interpretation. (This is because deductive arguments take their force only from the arrangement of the connectives and quantifiers in a sentence, rather than from the meanings of the nouns.) Consequently, without additional experimentation we have learned a great many new facts about points and great circles on the Earth; and in doing so we have used Euclid's work, even though when he did the work Euclid was not thinking about great circles, but about straight lines. Thus we see that we can often get new knowledge by taking old deductive arguments and interpreting the words in them in new ways. Modern mathematicians encourage such multiple interpretations by using words like "group", "ring", and so on which have no relevant everyday interpretations. At the same time, modern mathematicians leave their basic terms (like "set", "point", etc.) undefined. This is mostly because it is really impossible to define all words. (For example, we could define a "set" as a "collection of objects"; but then we would have to define "collection" and "object". We could define a "collection" to be an "aggregate" and an "object" to be a "thing"; but someone could ask what "aggregates" and "things" were. Clearly there is no end to this.) However, leaving our basic terms undefined also helps encourage multiple interpretations of the words.

For the reasons outlined above, modern mathematicians often present collections of mathematical knowledge in the following form. First there are some statements, written in terms of undefined words, which are not proven. These are called axioms. Then come the rest of the statements, together with deductive arguments showing that these statements are true under any interpretation of the undefined terms which makes the axioms true. These latter statements are called theorems, and the arguments are called proofs. (We will say more about proofs later.) The whole collection of axioms and theorems is called a theory, or a formal or axiomatic theory. This way of organizing and presenting mathematical results is called the axiomatic method.

Two views of mathematics. It is now possible to think of mathematics in two different ways. First, we can think of mathematics as the study of (abstract properties of) real objects. From this viewpoint the mathematical statement "2 + 2 = 4" would be interpreted as saying "When you put two objects with two other objects, you have four objects." Being statements about the real world, mathematical statements must be established by inductive arguments, or by deductive arguments whose premises were established by inductive arguments. Consequently we cannot be absolutely sure that mathematical "truths" are in fact true; although there is universal agreement on "2 + 2 = 4", there is often a difference of opinion regarding statements about the very small or the very large, e.g. "Every number can be divided by two" or "Infinite sets exist". Let us call mathematics as conceived in this way intuitive mathematics.

Mathematics can also be viewed as the construction of axiomatic theories. From this viewpoint "2 + 2 = 4" would be interpreted as saying, "If certain axioms are true under some interpretation of certain basic words, and if '2', '+', and '4' are defined in terms of the basic words in such-and-such a way, then '2 + 2 = 4' is true under that interpretation of the basic words"; or, more simply, just "'2 + 2 = 4' is a theorem of such-and-such an axiomatic theory" (the particular theory being fixed by the context in which this statement appears). Thus a mathematical statement merely says that there is a valid deductive argument leading from certain premises to a certain conclusion; we are not claiming that this conclusion is true, but only that it is provable from the (previously given) axioms, i.e. that it would be true if the axioms were true. Consequently we can be absolutely certain that mathematical "truths" are true, since they only assert the validity of deductive arguments, and the validity of a valid deductive argument can be checked beyond doubting. Call this system of mathematics formal mathematics.

To summarize: In intuitive mathematics, words have meanings (though these meanings cannot be rigorously and absolutely defined) and thus statements tell us something about reality; but truths are not absolute. In formal mathematics, words have no meanings (i.e. they are undefined) and statements tell us nothing about reality (but only about what reality would be like if we already knew certain things to be true; that is, formal mathematics is uncommitted about the true nature of reality); but truths are absolute. You may take your choice.

PROOFS

Nature of proof. In formal mathematics, a proof can be defined as a sequence of statements, each of which is the conclusion of a valid deductive argument all of whose premises are axioms, previous theorems, or earlier statements in the sequence. The last statement in the sequence is the theorem proved by the proof. It should be clear, for example, that if (A,B,C,D) is a proof of the theorem D, then (A,B,C) is a proof of the theorem C. In intuitive mathematics we can use the above definition with "axioms and previous theorems" replaced by "statements previously accepted as true".

Regardless of which view of mathematics we adopt, the techniques used in proofs are the same. The best way to learn what proofs are is to see lots of proofs. (Recall how you learned what a "horse" was.) The rest of this course will offer many opportunities for this. Many sorts of deductive arguments will be seen. We shall now describe some of the most commonly used ones. We shall talk in very general terms; specific examples will be seen in the rest of the course.

Proving "if...then" statements. Suppose we want to prove the statement "if A, then B". To do this, a mathematician might assume that "A" is true (that is, he adds "A" to his axioms or his collection of accepted truths) and then prove just "B". Having done this, he says that "if A, then B" has been proven. Why is this legitimate? What we have shown by our proof is that

> if you accept the axioms and "A" as all true,
> then you must accept "B" as true.

This is the same as saying that

> if you accept the axioms as true, and also
> accept "A" as true, then you must accept
> "B" as true.

This, in turn, says that

> if you accept the axioms as true, then you
> must admit that if you were to also accept
> "A" as true, then you would have to accept
> "B" as true.

But by the meaning of "if...then", this is to say that

> if you accept the axioms as true, then you
> must accept that "if A, then B" is true.

This discussion should convince you that if "B" can be proven from "A" and some other statements, then "if A, then B" can be proven from those other statements alone.

It is very important to realize, however, that the assumption in such proofs that "A" is true is only a *temporary* one. *At no time are we asserting that* "A" *actually is true*. In fact, "A" might actually be false (we know that "if A, then B" is true when "A" is false). The temporary assumption that "A" is true has *no connection* with the actual truth-value of "A", but is *just a device* used *inside* the proof of "if A, then B". The argument above shows that this device can produce convincing proofs. But after the proof is over, the assumption that "A" is true *must be discarded*. (Of course you can re-make the assumption in later proofs of "if A, then..." statements.)

Sometimes it is easier to prove "if (not B), then (not A)" by the above method than it is to prove "if A, then B". This is a legitimate and useful method, since the two quoted statements mean the same thing (as you showed in the discussion of the *contrapositive*).

Proving "and" statements. To prove "A and B", one must simply prove one of "A" and "B", and then prove the other. It may be possible to use "A" in the proof of "B"; this is all right (and may save time) *if* and *only if* you did *not* use "B" in the proof of "A". If neither of two statements is yet accepted as true, then proofs of both, each of which uses the other as a premise, will not be convincing. This error is called *circular reasoning*. It is easier to commit this error in intuitive mathematics than in formal mathematics, because in the former different workers may not agree on what has previously been accepted as true; one man might prove "A" using "B", another might prove "B" using "A", and a third might say (reading only the *results* of the first two) that together they had proved "A and B". But he would be wrong.

Proving "or" statements. Suppose we want to prove the statement "A or B". This can be done by proving just "A", or just "B", since "A or B" is true if either "A" or "B" is true.

But sometimes we may not see how to prove just "A", or just "B", directly. We can also prove "A or B" by proving "if (not A), then B", which can often be done by using the device discussed in the section on proving "if...then" statements. This method is legitimate since "A or B" *means the same thing* as "if (not A), then B", as you can (and should) show using truth tables.

Indirect proof. Suppose we were to construct a valid deductive argument, with "A", "B", and "C" as premises (their number is irrelevant), whose conclusion was known to be *false*. Then we would know that at least one of "A", "B", and "C" was false, since if they were all true the conclusion would have to be true. Suppose further that we know that all the premises other than "A" are in fact *true*. Then "A" *must be false*. Thus the given argument serves as a proof of the statement "not A". This method of proof is called *indirect proof*, *proof by contradiction*, or *reductio ad absurdum* (reduction to an absurdity).

Proof by cases. Suppose we have as our premise "A or B" and want to prove "C". If the premise is true, it means that either the situation described by "A" holds, or else the situation described by "B" holds (or both). The proof must convince people that "C" holds in *either case*. Thus the proof of "C" must consist of two parts; first, we must prove "C" from "A", and then we must prove "C" from "B". Then we can say: "Either 'A' or 'B' is true. If 'A' is true then the first argument proves 'C'. If 'B' is true then the second argument proves 'C'. Therefore 'C' is true." This is called a *proof by cases*.

If it is hard to prove "C" from "B" we can instead try to prove "C" from "B and (not A). If we succeed in this and have already proved "C" from "A", then we can say: "Either 'A' or 'B' is true. Also, either 'A' or 'not A' is true. If 'A' is true, then the first argument proves 'C'. If 'not A' is true, then 'A' is false, so 'B' is true (since one of 'A', 'B' is true); thus 'not A' and 'B' are both true and the second argument proves 'C'. So 'C' is true."

EXERCISES. 1. To say that the first proof-by-cases method described above is valid is to say that the statement

"If ((A or B) and (A → C) and (B → C)), then C"

is always true. Use truth tables to show that this statement is always true.

2. Describe the validity of the second proof-by-cases method given above by means of a single statement similar to that in Ex. 1, and show that it is always true.

3. Suppose "A or B" is true and that we can prove "C" from "A and (not B)" and also from "B and (not A)". Does this prove "C"?

4. Show that the statement "If ((A or B) and (not A)), then B" is always true and describe the method of proof suggested by it.

PROBABILITY AND STATISTICS

A supplementary unit for
Prospective Elementary Teachers

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PROBABILITY

I. INTRODUCTION

The weatherman says, "The probability of rain is about 1 out of 10". He could have said 1/10. If you had heard this report, would you be likely to carry an umbrella or perhaps wear a raincoat? Can you explain what the report means?

Purdue and Indiana both have good football teams. In a football game with Purdue, Indiana has a 50-50 chance of winning. Would you understand what we mean if we said, "The probability of Indiana winning is 1 out of 2 or 1/2"?

Riverboat Sam has a standard deck of 52 playing cards. He draws a card randomly from the deck. How likely is it that the card is red? How likely is it that it is a heart? How likely is it that it is the king of hearts? If Sam drew four more hearts out of the deck in succession, would you consider playing cards with Sam?

In the paragraphs above the word "probability" is mentioned several times. What does "probability" mean to you? Something uncertain! An element of chance! In other words, we talk about probability when an event could occur in more than one way, and no one could tell beforehand what will occur.

For example: Toss a coin. The coin could land heads or tails. You are not sure which will occur. However, the probability of it landing on tails is 1 out of 2, or 1/2. What is the probability of it landing on heads? **

Thus, we have the following

Definition: The probability of success of an event is defined as

$$p = \frac{\text{number of ways the event can succeed}}{\text{total number of ways the event can occur}} .$$

Exploration

1. Find the probability of getting a 4 on one toss of a die.
2. Find the probability of not getting a 4 on one toss of a die.

In the following four thought-provoking questions consider the set, $\{1,2,4,5,8,10,11\}$.

3. What is the probability of drawing an even number from this set?
4. What is the probability of drawing an odd number from this set?
5. What is the probability of drawing a number which is divisible by 3?
6. What is the probability of drawing a positive integer?
7. If p is the probability of success of an event and q is the probability of failure of that event, then what is the relationship between p and q?

II. EXPERIMENT **

We know that the probability of getting heads when tossing a coin is 1/2. Does this mean that when tossing 100 coins we will get exactly 50 heads?

1. Toss a coin 10 times and record the results in Table 1.

Toss	1	2	3	4	5	6	7	8	9	10
Heads										
Tails										
Ratio of heads										
Ratio of tails										
Percent of heads										
Percent of tails										

Table 1

2. Plot the cumulative percents of heads and of tails on the graph in Figure 1.

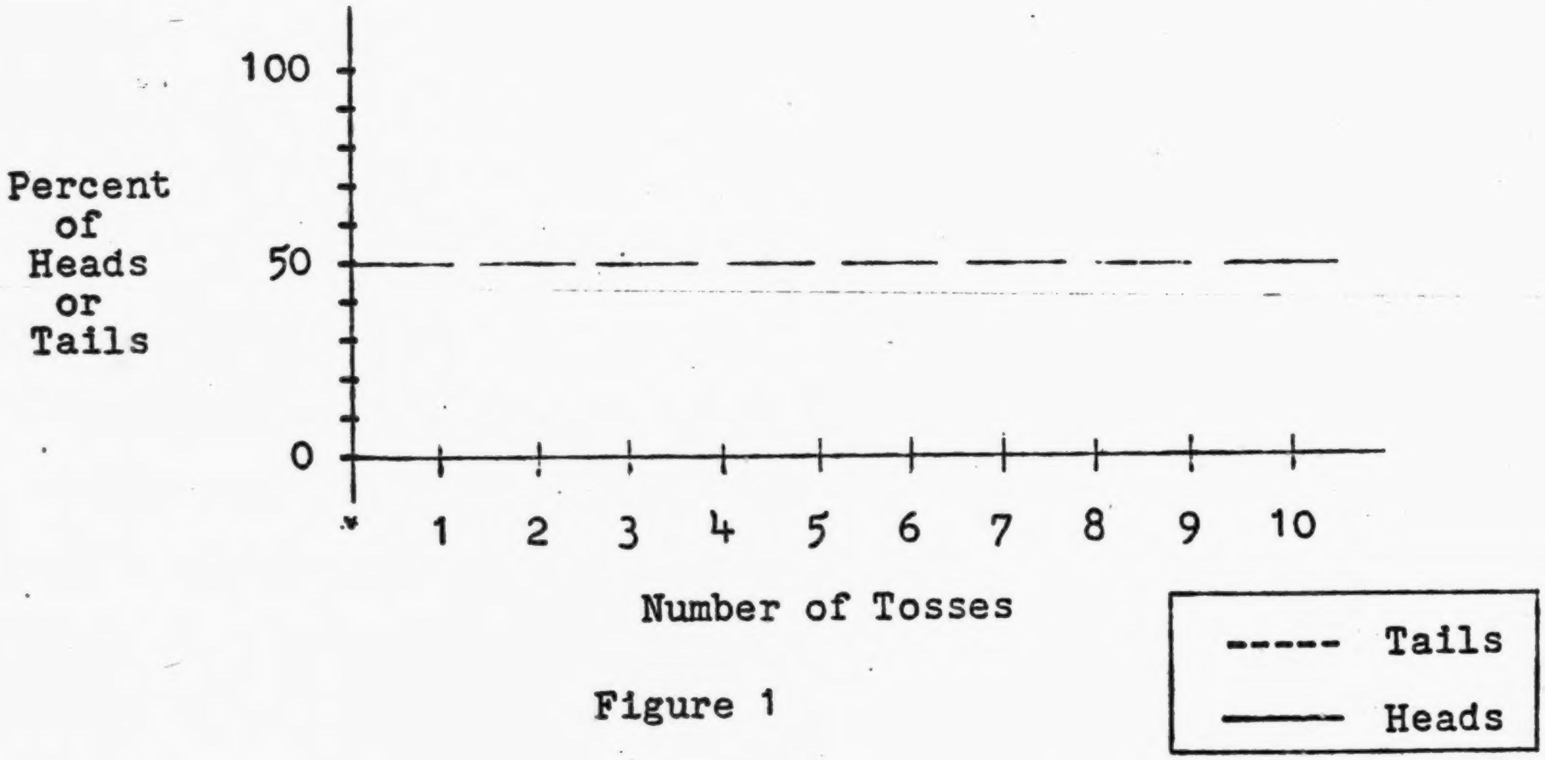


Figure 1

3. What do you notice about the percent of heads obtained as the number of tosses increases?

4. What do you notice about the percent of tails obtained

as the number of tosses increases?

5. What do you think would happen in the graph if you tossed the coin 100 times?   1000 times?
6. Then what do we mean when we say the probability of getting heads is 1/2 "in the long run"?

III. SAMPLE SPACES

1. Write down the set of all the possible outcomes of throwing two dice. List the outcomes as ordered pairs in which the first component is the number on the first die and the second component is the number on the second die. Note that each outcome is equally likely.

$$\left\{\begin{array}{llllll}
(\;,\;) & (\;,\;) & (\;,\;) & (\;,\;) & (\;,\;) & (\;,\;) \\
(\;,\;) & (\;,\;) & & & & \\
(\;,\;) & (\;,\;) & & & & \\
(\;,\;) & & & & & \\
(\;,\;) & & & & & \\
(\;,\;) & & & & &
\end{array}\right\}$$

2. How many different ordered pairs did you get?
3. How do you know there are no other outcomes? **
4. Plot these outcomes on the graph on the next page. Put a dot for each ordered pair.

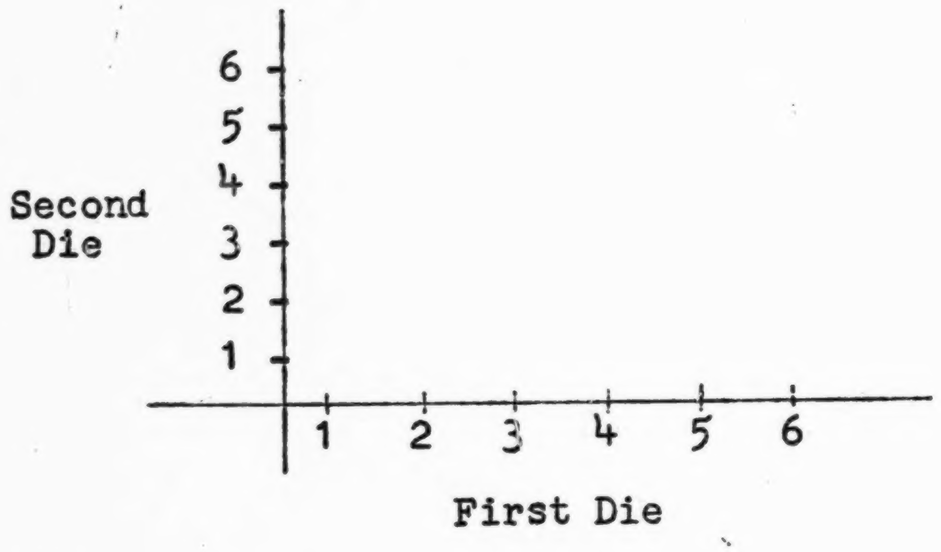


Figure 2

5. You have just drawn a picture of the SAMPLE SPACE for throwing two dice. Thus the sample space is a representation of all possible ways an event can occur. The sample space is used to aid in determining the probability for an event to succeed.
6. What is the sample space for the sums obtained when tossing two dice? Is each outcome equally likely?
7. How many ways can you get a sum of 8 when tossing two dice?
8. What is the probability of getting a sum of 8 when tossing two dice?

Exercises

1. What is the probability that the next person you meet was not born on a Sunday? **
2. List the sample space for tossing three coins. The outcomes will be ordered triples. Find the probability of getting
    a. exactly 2 heads.

b. no heads.

c. at least one head.

d. at most one tail.

3. Mark each statement as true or false.

T F a. A student can either pass or fail a course. Therefore the probability of passing is 1/2.

T F b. When tossing 2 dice, the probability of getting a sum of 9 is the same as the probability of getting a sum of 5.

T F c. In a room containing 30 people, the probability that at least two persons in the room have the same birthday is greater than 1/2.

4. Consider drawing 2 cards from a deck consisting of the 2,3,4,...,10 of hearts.

a. What is the probability that the product of the numbers on the cards is even?

b. What is the probability that the product of the numbers on the cards is odd?

c. Do parts a. and b. considering the sum of the numbers on the cards as being even; being odd.

STATISTICS

I. MEASURES OF CENTRAL TENDENCY

Consider 1.

Listed below are the salaries of a random sample of persons employed by Utopian State University.

\$4000	Janitors
\$5000	
\$7000	Professors
\$8000	
\$10,000	
\$11,000	
\$12,000	
\$14,000	
\$17,000	Dean
\$18,000	President
\$75,000	Football Coach

What would you consider to be a realistic "average" of the salaries paid to persons employed at U.S.U.? In other words, what single number would most accurately represent the salary of an employee of U.S.U.? **

Consider 2.

Janet Schmalz received the following grades on her mathematics tests: 60, 60, 60, 90, 98, 100 .
What would you estimate as her "average" grade? Again, what single number would most accurately represent her grade in the course? **

Consider 3.

The class sizes at Sweet William College are as follows:

2,3,3,7,8,8,12,13,25,25,25,25,25,175 .

What would be considered a valid approximation of the "average" class size? **

II. MEASURES OF DISPERSION

Consider the following sets of scores:

$$A = \{ 4,6,8,10,12,14,16 \}$$

$$B = \{ 4,7,9,10,11,13,16 \}$$

The scores in A range from 4 to 16. Likewise, the scores in B range from 4 to 16. So we say that the range for both A and B is 12. How is this found? Note the following:

1. The mean of scores in A is 10.
2. The mean of scores in B is 10.
3. The median for A is 10.
4. The median for B is 10.

But how do these set of scores differ? Notice that the scores in B seem to be more closely clustered around 10 than those in A. We will use this fact to distinguish between A and B.

Since 14 is above the mean ($\overline{X} = 10$) for both A and B, 14 is a positive deviation from the mean. With 6 below the mean, then 6 is a negative deviation from the mean. Thus the deviation of a score from the mean is either positive or negative or zero.

For example:

Score X	Deviation from the Mean $X - \overline{X}$
4	-6
6	-4
8	-2
10	0
12	2
14	4
16	6
	0

$\overline{X}$ = Mean of the Scores

If we add all the deviations together we will get 0. Why? This is not very useful. So by squaring each deviation, each squared deviation becomes a non-negative number, and then dividing by the number of scores, we find an average squared deviation, which is called the variance of the set of scores. In order to apply this measure in practical situations we will find the square root of the variance. This measure is called the standard deviation of a set of scores. The standard deviation is represented by s.

$\overline{X}$ is the mean

X_i is the $i^{\underline{th}}$ score in the list of scores

$\sum$ is a summation symbol

n is the number of scores

$$s = \sqrt{\frac{\sum_{i=1}^{n} (X_i - \overline{X})^2}{n}}$$

EXAMPLE 1

Consider the set of scores:

$$A = \{ 4,6,8,10,12,14,16 \}$$

$$n = 7$$

$$\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n} = \frac{70}{7} = 10$$

X_i	$(X_i - \overline{X})$	$(X_i - \overline{X})^2$
4	-6	36
6	-4	16
8	-2	4
10	0	0
12	+2	4
14	+4	16
16	+6	36
$\sum_{i=1}^{7} X_i = 70$	$\sum_{i=1}^{7}(X_i - \overline{X}) = 0$	$\sum_{i=1}^{7}(X_i-\overline{X})^2 = 112$

so

$$s = \sqrt{\frac{\sum_{i=1}^{7} (X_i - \overline{X})^2}{n}}$$

$$= \sqrt{\frac{112}{7}}$$

$$= \sqrt{16}$$

$$= 4$$

EXAMPLE 2

Consider the set of scores:

$$B = \{ 4,7,9,10,11,13,16 \}$$

Here again the mean is 10 and n is 7.

X_i	$(X_i - \overline{X})$	$(X_i - \overline{X})^2$
4	-6	36
7	-3	9
9	-1	1
10	0	0
11	1	1
13	3	9
16	6	36
$\sum_{i=1}^{7} X_i = 70$	$\sum_{i=1}^{7}(X_i-\overline{X}) = 0$	$\sum_{i=1}^{7}(X_i-\overline{X})^2 = 92$

so

$$s = \sqrt{\frac{\sum_{i=1}^{7} (X_i - \overline{X})^2}{n}}$$

$$= \sqrt{\frac{92}{7}}$$

$$= \sqrt{13.14}$$

$$= 3.62$$

**

Exercises

Find the mean, median, mode, range and standard deviation of the following sets of numbers.

1. $\{ 4,8,10,10,12,16 \}$

2. { 2,3,5,2,1,5,7,1,2,6,11 }

3. { 954,947,943,951,949,951,946,943,945,951 }

(Hint for #3: Try to find any easier way to obtain the mean.) **

III. NORMAL DISTRIBUTION

Consider the following IQ scores from Upper Gooseneck Regional High School.

78	91	96	100	104	108	119
82	92	97	101	105	110	123
84	92	98	102	105	111	
86	94	98	102	106	113	
87	95	100	102	107	114	
89	96	100	103	107	116	

Let us now group these scores into intervals with a length of 5.

Interval	Frequency of scores
76- 80	1
81- 85	2
86- 90	3
91- 95	5
96-100	8
101-105	8
106-110	5
111-115	3
116-120	2
121-125	1

Now put these values on a graph by placing a dot on the graph for each ordered pair, (a,b), where a is the interval and b is the number of scores in that interval. (Figure 3, page 13)

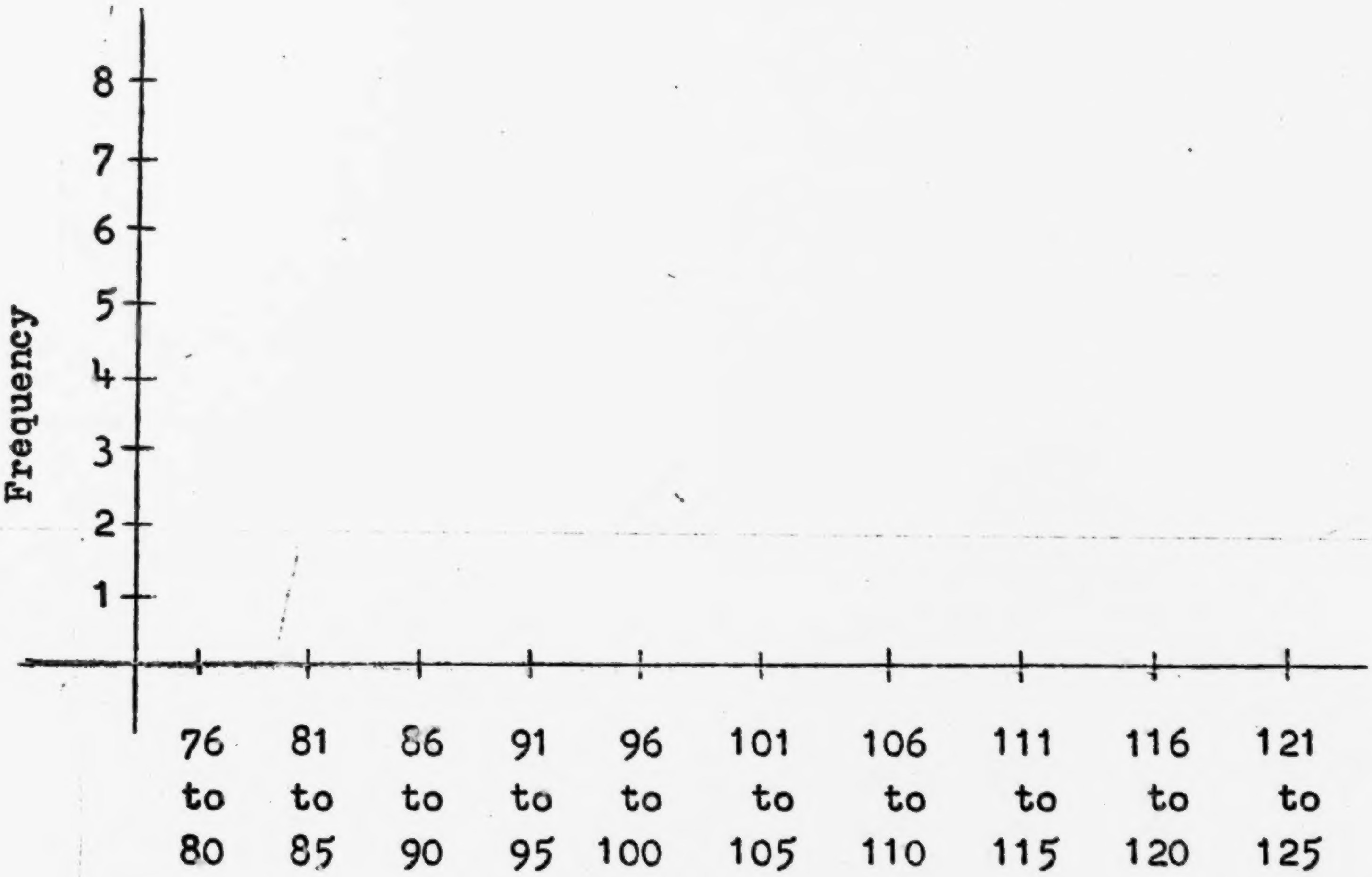
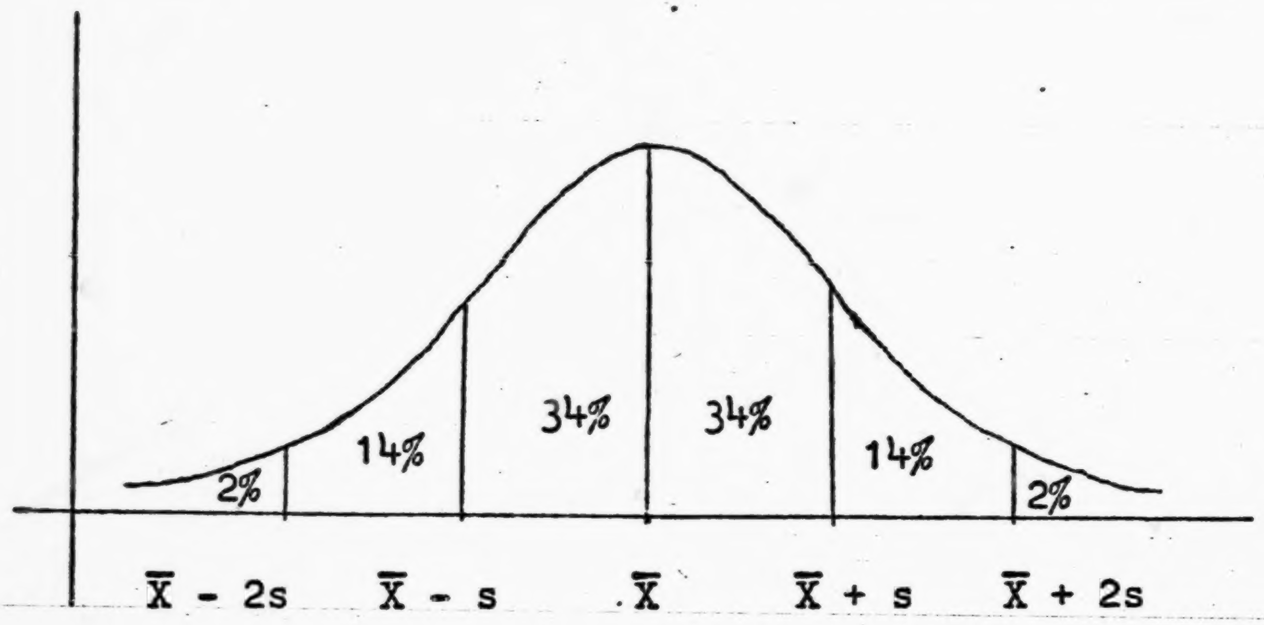


Figure 3

Now connect the points on the graph with a smooth curve. Notice that the curve is shaped like a bell. For this reason it is sometimes called "a bell-shaped curve."

Intelligence is one of the characteristics of human being which is "distributed" in this way. Other characteristics which tend to be "normally distributed" are height and weight for human beings of the same sex.

Let us consider the graph below as a graph of a normally distributed characteristic.



The mean of this distribution is designated by $\overline{X}$. The standard deviation is given by s. The percentages given on the graph indicate the percent of persons one would expect to score in the indicated range. This says, for example, that 68% of the population would be expected to achieve scores within one standard deviation above or below the mean. It also indicates that 96% would be expected to achieve scores within two standard deviations below or above the mean.

EXAMPLE

On an IQ test the mean is 100 and the standard deviation is 10. Therefore, we would expect about 68% of the scores to lie between 90 and 110. We would expect only 2% of the scores to lie below 80. What percent of the scores would you expect to find between 80 and 110? (ans. 82%)

IV. CORRELATION

You as a teacher have administered two tests. The students and their scores are listed below.

Student	First test	Second test
Abe	75	92
Bob	87	85
Conrad	77	95
David	92	77
Elmer	80	75
Frank	82	97
Gerald	95	82
Homer	97	80
Ike	90	90
Jerome	85	87

Note that although the test scores for the class are identical, the scores for individual students are not identical except in one case. Thus, the mean and median on the first test are the same as the mean and the median on the second test. Can you conclude that the tests were equally difficult? To find how the scores on first test are related to the scores on the second test we use a coefficient of correlation.

There are several kinds of coefficients of correlation. The one we will consider is called the coefficient of rank correlation.

This number r is defined in the following way:

$$r = 1 - \frac{6\Sigma d^2}{n(n^2-1)} ,$$

where one assigns ranks to the scores on both tests and then finds the differences (d) in the ranks for each student, as follows:

	SCORES		RANKS			
Student	1st test	2nd test	1st test	2nd test	d_i	d_i^2
A	75	92	10	3	7	49
B	87	85	5	6	-1	1
C	77	95	9	2	7	49
D	92	77	3	9	-6	36
E	80	75	8	10	-2	4
F	82	97	7	1	6	36
G	95	82	2	7	-5	25
H	97	80	1	8	-7	49
I	90	90	4	4	0	0
J	85	87	6	5	1	1

$\Sigma d^2 = 250$ $n = 10$, the number of students

$$n(n^2 - 1) = 10(100 - 1) = 990$$

$$r = 1 - \frac{6 \sum_{i=1}^{10} d_i^2}{n(n^2-1)}$$

$$= 1 - \frac{6 \cdot 250}{990}$$

$$= 1 - 1.5$$

$$= -0.5$$

Notice that students who ranked high on the first test tended to get low scores on the second and that students who ranked low on the first test tended to receive high scores on the second test. This is also indicated by the negative value of r . If students who score high on the first test also score high on the second test and those who score low on the first also score low on the second test, then r will be a positive number. The highest possible value for r is 1.0; this indicates that each student had the same rank on both tests. The lowest value for r is -1.0. **

Exercises

1. Suppose you know that a distribution of scores is normal and you know that the mean is 75. What is the median? What is the mode?

2. The mean on an IQ test is 100 and the standard deviation is 15. What percentage of the population would be expected to have IQ scores above?

   a. 130
   b. 115
   c. 100
   d. 85
   e. 70

What percent of the population could be expected to have IQ scores between:

f. 85 and 115
g. 70 and 130
h. 85 and 130

3. On a standardized achievement test the mean is 62 and the standard deviation is 7. Two-thirds of the persons that took this test should have achieved scores in what range? You would expect 2% to achieve above what score?

4. On a test which you have constructed for your class, which has a normal distribution, the mean is 50. You decide to assign grades on the following basis:

A - more than 1½ s.d. above the mean

B - ½ s.d. to 1½ s.d. above the mean

C - ½ s.d. below the mean to ½ s.d. above the mean

D - 1½ s.d. to ½ s.d. below the mean

F - less than 1½ s.d. below the mean

It turns out that the lowest score for an A is 77. Determine the range of scores for B, C, D, and F.

A - 77 or above

B - ___ to ___

C - ___ to ___

D - ___ to ___

F - ___ or below

5. The following chemistry students at Hardtimes Pyrotechnic Institute were ranked by their lecture instructor and their laboratory assistant as follows:

Student	Lecture	Lab
A	9	8
B	5	3
C	10	9
D	1	2
E	8	7
F	7	10
G	3	4
H	4	6
I	2	1
J	6	5

How do these rank assignments correlate with each other? What does this number mean?

6. You have just given two tests in arithmetic to your class of twenty students. Find the rank correlation between the two tests. What does this number mean?

	SCORES		RANKS			
Students	Test 1	Test 2	Test 1	Test 2	d	d^2
A	77	60				
B	93	65				
C	84	76				
D	82	71				
E	91	69				
F	75	58				
G	98	73				
H	61	40				
I	100	75				
J	54	29				
K	89	67				
L	65	42				
M	95	72				
N	71	50				
O	96	77				
P	79	55				
Q	46	33				
R	80	63				
S	74	51				
T	86	62				

COMMENTARY FOR INSTRUCTORS

Section 1

1. Bring out that probability ranges from 0 to 1. Thus 0-impossible event 1-sure event. You might use sets and subsets to develop concept. Give many more examples. If you wish to use sets, you could develop it it in the following way. Looking at this definition of probability, we can relate it to set notation.

U universal set - Sample space

A a subset of U - all ways event succeeds

For example, let U be the set of all American coins

$U = \{ \text{penny}, \text{nickel}, \text{dime}, \text{quarter}, \text{half-dollar} \}$

The event $A = \{$ any American coin whose value is greater than 20 cents $\}$

$= \{ \text{quarter}, \text{half-dollar} \}$

then $p(A)$ = probability of A being a success event

$$= \frac{2}{5}$$

$$= \frac{n(A)}{n(U)}$$

If we let event $B = \left\{ \begin{array}{l} \{ \text{all American coins whose value} \\ \text{is greater than zero} \} = \{ \text{penny}, \\ \text{nickel}, \text{dime}, \text{quarter}, \text{half-quarter} \} \end{array} \right\}$

Then $p(B) = \frac{5}{5} = 1$. This is the largest value p can take.

Or $A = U$ Sure Event

If event $D = \{$ all American coins whose value is greater than 60 cents $\}$

$= \emptyset$

$p(D) = \frac{n(D)}{n(U)} = \frac{0}{5} = 0$ This is an impossible event.

$A = \emptyset$. Thus, values of p can range from 0 to 1.

If you know what A is, what about A' or when A fails?

$$A \cup A' = U \quad \text{so} \quad P(A \cup A') = P(U)$$

But $P(A \cup A') = 1$ for $A \cap A' = \emptyset$.

Event A' = { an American coin whose value is equal to or less than 20 cents }

= { penny, nickel, dime }

$p(A') = 3/5$ Thus $P(A) + P(A') = 1$

or $p(A) + p(A') = p(A \cup A') = 1$

Thus $P(A') = 1 - P(A)$

Could show by Venn diagrams.

Then, look at these ideas with Venn diagrams.

2. Do this experiment in class to show the Law of Large Numbers-which states informally, by choosing the sample size n sufficiently large, the probability that the value of the sample mean differs from the population mean by at most c can be made as small as or as close to 1 as we like.

An example of table 1 is below.

Toss	1	2	3	4	5	6	7	8	9	10
Heads	0	1	0	0	1	1	1	0	1	0
Tails	1	0	1	1	0	0	0	1	0	1
Ratio of heads	0/1	1/2	1/3	1/4	2/5	3/6	4/7	4/8	5/9	5/10
Ratio of tails	1/1	1/2	2/3	3/4	3/5	3/6	3/7	4/8	4/9	5/10
Percent of heads	0	50	33	25	20	50	57	50	55	50
Percent of tails	100	50	67	75	80	50	43	50	45	50

Thus as the number of tosses increase the cumulative percentage of heads will approach $\frac{1}{2}$. However, stress one does not always get exactly one half number of heads in any specific number of tosses.

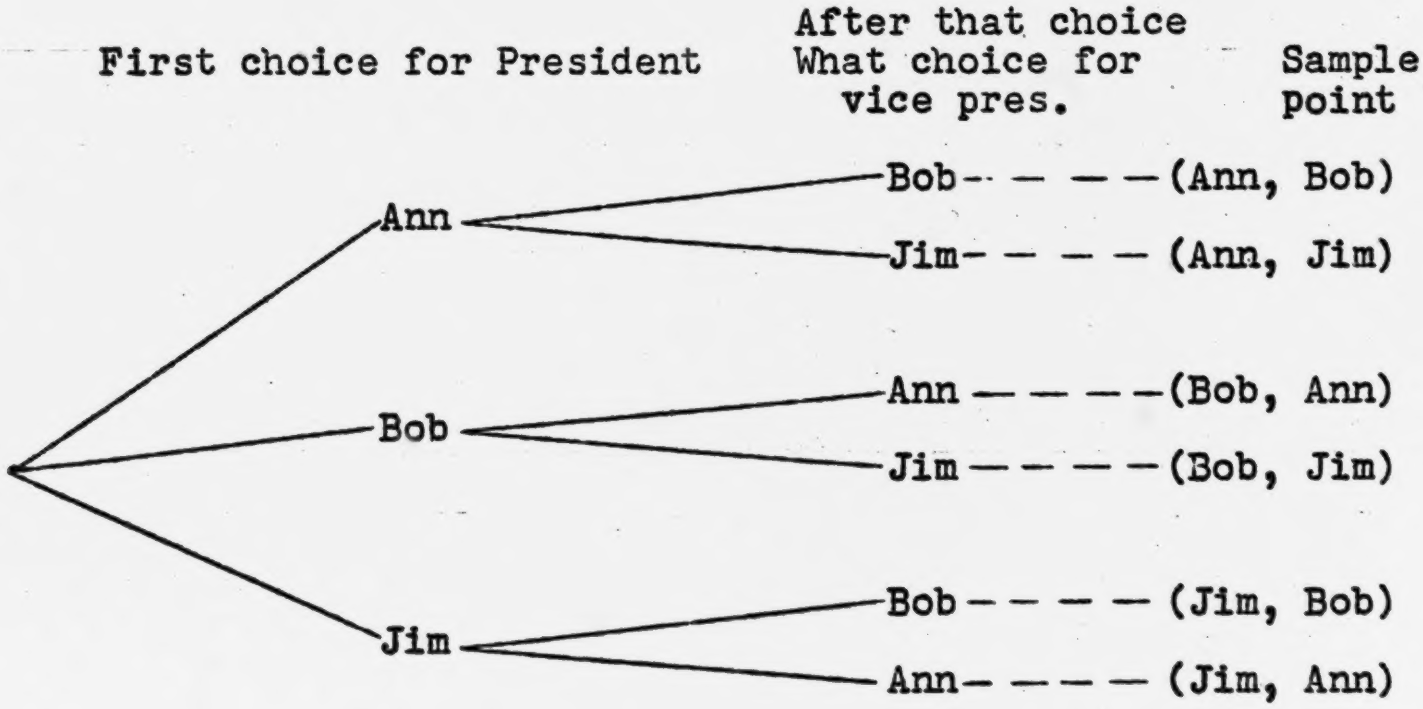
3. Make students aware of the fact that by listing the sample space, outcomes will not be excluded. Always ask how can this event happen. Also at this time use the tree diagram or position blanks to develop a counting procedure.

Example 1.

Event: Electing a president and vice president out of the set { Ann,Bob,Jim }

Sucessful event: Selecting a boy for President and a girl for vice president.

By branching (tree diagram):


First choice for President
After that choice What choice for vice pres.
Sample point
Ann
Bob
Jim
Bob
Jim
Ann
Jim
Bob
Ann
(Ann, Bob)
(Ann, Jim)
(Bob, Ann)
(Bob, Jim)
(Jim, Bob)
(Jim, Ann)

By position:

number of choices for pres. ____________ then how many for vice pres. ______________

Either way we get six different outcomes. Branching is a good beginning, then use positions.

So the counting principle states that if there are m ways to do the first thing and n ways to the second thing and r ways to do the third thing, then the total number of different ways that the event can occur is $m \cdot n \cdot r$.

$$\underline{\quad m \quad}\ \ \underline{\quad n \quad}\ \ \underline{\quad r \quad} = mnr$$

Thus, successful event is $\underset{\substack{\text{pres.}\\ \text{boy}}}{\underline{\quad 2 \quad}}\ \ \underset{\substack{\text{vice}\\ \text{girl}}}{\underline{\quad 1 \quad}} = 2 \cdot 1 = 2$

Then, P(successful event) = 2/6.

Example 2.

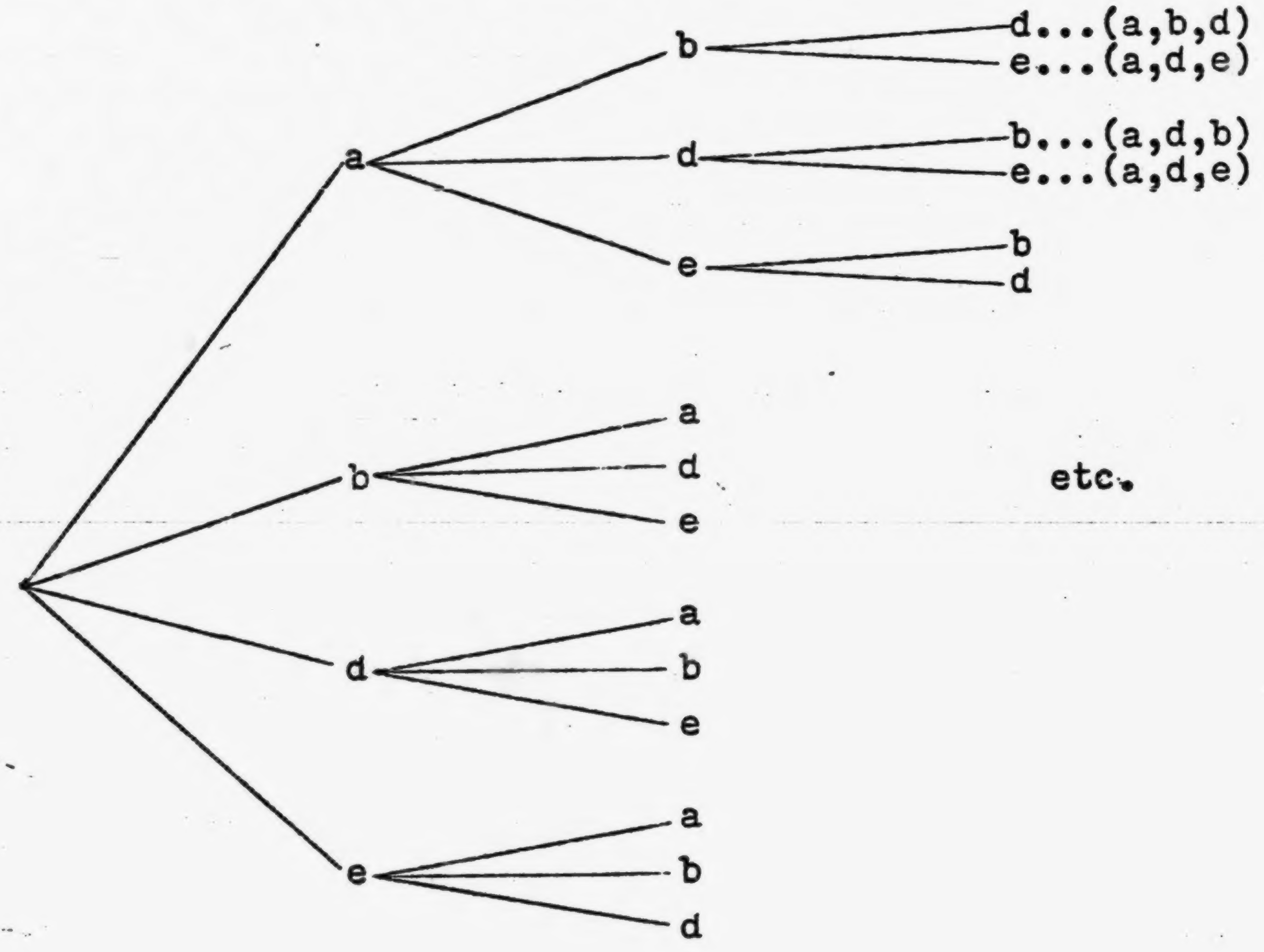
Event: Forming three letter words from { a,b,d,e },

i.e., ade,bea.

Successful event: All words beginning with a

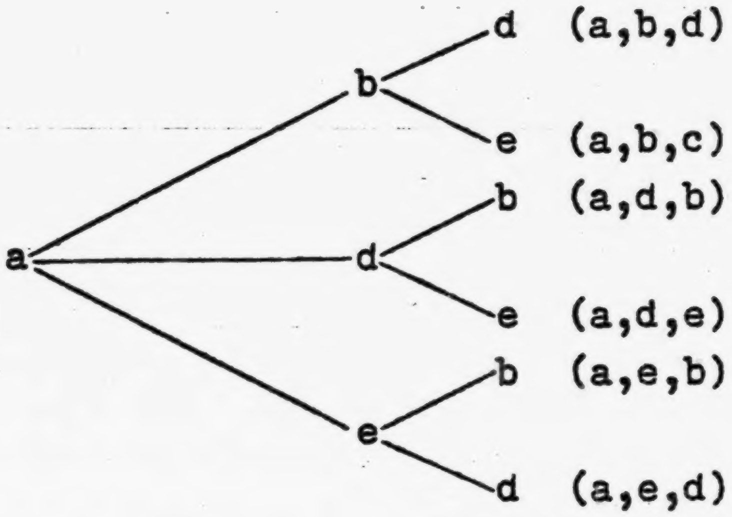
Event: $\underset{1^{st}}{\underline{\qquad}}\ \ \underset{2^{nd}}{\underline{\qquad}}\ \ \underset{3^{rd}}{\underline{\qquad}} \rightarrow \underline{\quad 4 \quad}\ \ \underline{\quad 3 \quad}\ \ \underline{\quad 2 \quad}$ 14 total words

letter


d...(a,b,d)
e...(a,d,e)
b...(a,d,b)
e...(a,d,e)
b
d
a
b
d
e
b
a
d
e
etc.
d
a
b
e
e
a
b
d

Successful Event: _1_ _3_ _2_ = 6

or


d (a,b,d)
b
e (a,b,c)
b (a,d,b)
a
d
e (a,d,e)
b (a,e,b)
e
d (a,e,d)

You could also work out different telephone numbers or license plate numbers.

4. Assume each day is equally likely. $p = 6/7$

On question 3a you don't need to know the correct answer but imply that the two events are not equally likely; many things determine the probability.

On exercise 3c, same birthday means same day of year, not exact birth date, i.e., May 9. True

To justify the above question consider the following reasoning:

With r-people Sample space would have 365^r elements

$\underline{365}\quad \underline{365}\qquad \underline{365}\ \ldots .\ \underline{365}$ r blanks or 365^r

For successful event: at least two among the r people have the same birthday. Easier to look at E' where

$n(E) + n(E') = 365^r$

E': number of ways of selecting r different birthday

$\underline{365}\quad \underline{364}\quad \underline{363}\quad \underline{362}\quad \underline{361}\quad \underline{360}\quad \ldots\quad \underline{365 - r + 1}$

so $n(E) = 365^r - (365 \times 364 \times 363 \times \ldots \times 365 - r + 1)$

So $P(E) = \dfrac{365^r - (365 \times 364 \times 363 \times \ldots \times 365 - r + 1)}{365^r}$

So when r =

r	10	20	22	23	24	30	40	50
P(E)	.12	.41	.48	.51	.54	.71	.89	.97

Section 2

5. The student should choose a number close to 11,000 which is the median or middle score of the distribution. Define and write on board the median

as the middle score, after the students have worked on the exercise. Note that median is not affected by extreme scores as the other kinds of averages. Also a quick estimate.

6. The student should choose a number close to 78 which is the mean. Define and write on board: Mean is the sum of the scores divided by the number of scores. After the student has completed the exercise, note that the median would not be an appropriate average for these scores.

7. The student should choose 25 which is the mode. Define and write on the board. Mode: the most frequent occurring number in the distribution. Why would the mode be the most appropriate here?

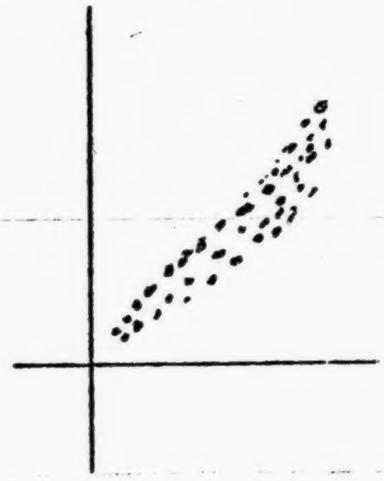
8. Be sure to go through examples with students. On exercise 3, by subtracting 943 or any other score from each score, averaging the converted scores then adding 943 or the score you choose on to the converted average, one obtains the mean of these scores.

Score	X-943
954	11
947	4
943	0
951	8
949	6
951	8
946	3
943	0
945	2
951	8
	50

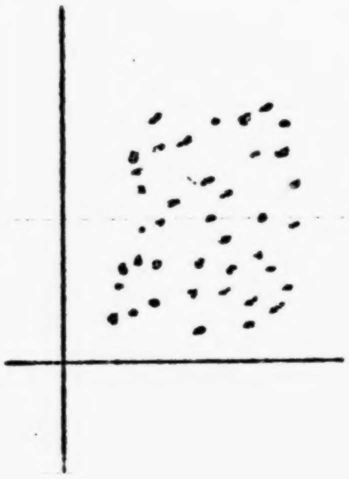
$\bar{X} = \frac{50}{10} = 5$ Then $\bar{X} = 5 + 943 = 948$

Section 3

9. Also show correlation by scatter diagrams as discussed below.



r is close to +1



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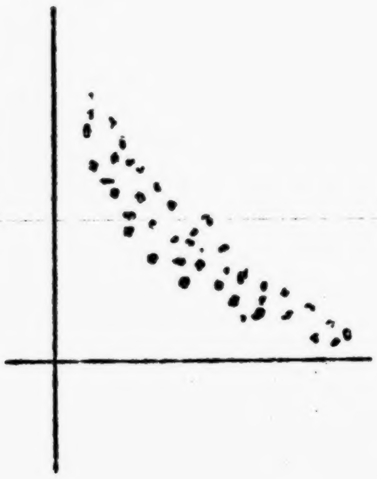

r is close to -1

COURSE OUTLINE - T106

Required Textbooks

S Smart, J. R., *Introductory Geometry: An Informal Approach*, Brooks-Cole Publishing Co., Belmont, California, 1967.

Y National Council of Teachers of Mathematics, *Enrichment Mathematics for the Grades, Twenty-Seventh Yearbook*, National Council of Teachers of Mathematics, Washington, D. C., 1963.

Optional Textbooks (available in Swain Hall Library)

A Adler, C. F., *Modern Geometry: An Integrated First Course*, McGraw Hill Book Co., Inc., New York, 1958.

B Brumfiel, C. F., Eicholz, R. R., and Shanks, M. E., *Geometry*, Addison-Wesley Publishing Co., Inc., Reading, Massachusetts, 1960.

C Coxeter, H. S. M., *Introduction to Geometry*, John Wiley and Sons, Inc., New York, 1961.

E Eves, Howard, *An Introduction to the History of Mathematics*, Holt, Rinehart, and Winston, Inc., New York, 1953.

K Keedy, M. L. and Nelson, C. W., *Geometry, A Modern Introduction*, Addison-Wesley Publishing Co., Inc., Reading, Massachusetts, 1965.

M Meserve, B. E. and Sobel, M. A., *Introduction to Mathematics*, Prentice-Hall Inc., Englewood Cliffs, N. J., 1964.

N Newman, J. R., *The World of Mathematics*, Simon and Schuster, New York, 1956.

R Ringenberg, L. A., *Informal Geometry*, John Wiley and Sons, Inc., New York, 1967.

To The Student

In recent years there has been a revolution in school mathematics. A quick glance into some newer elementary school textbooks will show that skill in computation is no longer sufficient criteria for teaching elementary school mathematics. The sequence of courses consisting of T104, T106, T108 is designed to give prospective elementary teachers the background in mathematics that they must have in order to teach mathematics successfully to elementary school pupils.

The courses T104, T106, T108 are concerned with mathematical ideas of two types: those which will be taught to elementary pupils and those which will give the prospective teacher a deeper understanding of the concepts he will teach.

These courses are strictly mathematical in nature; a companion course, E343, will instruct the prospective teacher in the techniques of the actual presentation of the mathematical ideas to her pupils.

In order that the students may benefit maximally from the lectures, it is strongly recommended they study beforehand the material to be presented in each lecture.

The texts required for T104 will also be used in this course. With the additional required texts for T106, the student will have the foundation of a reference library which will probably be of great use while teaching.

Course Content

1. Introduction
    - A. Nature of Geometry
        - S: pp . 1-3
        - R: 1.1
        - A: pp. 3-5
        - B: 1.1
        - Y: Chapter 9
    - B. Origin of Geometry
        - ~~S: pp. 4-~~5
        - R: 1.2
        - B: 1.2
        - E: 2.4, 2.9, 3.2, 4.1 - 4.4
    - C. Informal approach to Geometry
        - S: pp. 3-5
    - D. Logic
        - Y: pp. 291-301
2. Basic Concepts
    - A. Review of Sets
    - B. Points
    - C. Angles
    - D. Lines and Planes
    - E. Parallelism
        - S: Chapter 2
        - R: 2.1
        - K: 3.4
3. Measurement of line segments and angles
    - S: Chapter 3
    - R: 2.2, 2.3
4. Simple Closed Curves, Polygons, and Polyhedra
    - A. Simple Closed Curves
        - S: Chapter 4
        - R: 2.1, 5.1
        - K: P. 43-45
    - B. Quadrilaterals
        - S: Chapter 4
        - R: 2.2, 3.1, 3.2, 3.5, 5.3

4. Simple Closed Curves, Polygons, and Polyhedra (cont.)
   - C. Parallelism
     R: 3.1 - 3.5
   - D. Additional Topics
     K: Theorem 6.7

5. Circles
   S: Chapter 5

6. Geometric Constructions
   S: Chapter 6 R: 12.1 - 12.5

7. Congruence
   - A. Triangle Congruence
   - B. General Concept of Congruence
     S: Chapter 7
     R: 2.3, 2.4, p. 32
     K: 4.4
     Math Teacher, April 1968
   - C. Similarity
     S: p. 94
     R: 2.2 - 2.3

8. Area of Plane Figures
   S: Chapter 8
   R: 4.1 - 4.7

9. Errors in Measurement
   R: 2.8
   S: Chapter 9

10. Similarity
    S: Chapter 10
    R: 2.6, 2.7

11. Algebraic Representation of Geometric Figures
    S: Chapter 11 (omit trigonometric ratios)
    M: Chapter 8

12. Geometry of Selected Space Figures
    - A. Theorems on Lines and Planes
      S: Chapter 12
      R: 5.2 - 5.4
    - B. Dihedral and Polyhedral Angles
      S: Chapter 12
      R: 5.5
    - C. Polyhedra
      S: Chapter 12
      R: 6.1 - 6.4
      K: 4.12

13. Locating points in space
    S: Chapter 13 (omit coordinates for points in space)

14. Volume and Surface Area
    S: Chapter 14
    R: 6.5 - 6.13

15. Formal, Informal, and Experimental Geometry
    S: Chapter 15
    R: 1.3
    K: Chapter 1

16. Modern Geometry
    A. Topology
    B. Finite Geometry
        M: 10.5
    C. Non-Euclidean Geometry
        S: Chapter 16
        M: 10.1 - 10.3

COURSE OUTLINE

AND

TEACHING GUIDE

T106

Geometry for Elementary Teachers

August 1968

The following time outline has been included so that the teaching associate can plan his teaching effectively. We recommend that the teaching associate follow the sequence shown, since much of the content depends on preceding content.

CHAPTER	NUMBER OF DAYS
1	1
2	2
Quiz and Review	1
3	2
4	3
Test #1	1
5	3
6	2
Quiz and Review	1
7	3
Unit #1	2
8	2
9	2
Test #2	1
10	3
11	2
Quiz and Review	1
12	3
13	1
Test #3	1
14	3
15	2
16	3

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I. Introduction

Objectives of the Course

1. To help prospective elementary teachers develop an understanding of the fundamental ideas which are the core of geometric knowledge for all who have occasion to work with mathematics. This includes such concepts as congruence, measurement, parallelism, and similarity.

2. To develop geometric intuition and insight. To this end, the students should be allowed to participate in the formulation of axioms which are based on intuitive plausibility arguments.

3. To develop an appreciation of a deductive system. While no single body of axioms is developed for all of Euclidean geometry, several deductive subsystems should be formulated and should be developed. In addition, several other deductive systems should be examined. If it is not possible to elaborate upon the details in the classroom, the student should be assigned the task of constructing the necessary arguments. The formulation of such arguments is a necessary part of any mathematics course.

4. To develop the ability to apply geometric ideas and to identify some practical applications of geometry. Of the several mathematics courses proposed for the elementary education curriculum, geometry perhaps lends itself best to exercise in problem solving and to the development of techniques of problem solving; full advantage of this should be taken.

5. To show that certain geometric systems can, with varying degrees of accuracy, describe properties of the physical world.

II. Course Outline

Note: Added references are background material to be used at the discretion of the TA.

1. Introduction

A. Nature of Geometry

The student should be shown the difference between geometry as the study of physical space and geometry as a deductive system. Thus discuss why geometry is studied and some of the practical applications of it.

References
S: pp. 1-3
R: 1.1
A: pp. 3-5 Good introduction ideas
B: 1.1
Y: Chapter 9 Gives guide lines for course

B. Origin of Geometry

Discuss some of the more important aspects of the history of geometry including the contributions of Euclid, Pythagoras and some of the modern geometers. In addition to the references provided here much information can be found in books on the history of mathematics.

References
S: pp. 4-5
R: 1.2
B: 1.2 Good introduction
E: 2.4, 2.9, 3.2, 4.1 - 4.4

C. Informal approach to Geometry

Look at the framework of geometry to develop a frame of reference in studying geometry.

References
S: pp. 3-5

D. Logic (for students who had T104 before September 1967)

Discuss the nature of proof and proof procedures.

References
Y: pp. 291-301

2. Basic Concepts

The primary purpose of this section is to introduce or reinforce the student's knowledge of the language of geometry, some of the basic concepts relating to geometric figures, and the relationships between geometric figures. Definitions of terms need not be made absolutely precise at this time and there is no need to prove theorems in this section. More rigorous definitions and proofs of theorems will be provided later. An effort should be made to lead the students to state definitions and properties in their own words. The "discovery" approach should be used extensively in teaching this section.

A. Review of Sets

Develop the relationship between set notation and geometric terms in order to have more precise definitions.

B. Points

The concept of a point should be introduced as a position or a location. Thus sets of points are considered which lead to the definitions of space as the set of all points.

C. Angles

Define an angle as the union of two distinct rays with a common endpoint. Then investigate vertical angles, adjacent angles, separation of a plane by an angle, and the interior and exterior of an angle.

D. Lines and Planes

A plane and a line are subsets of space. We can define a line as the intersection of two planes and a point as the intersection of two lines. With this in mind, consider open and closed segments, rays, and half lines as subsets of a line.

E. Parallelism

Two lines in the same plane are parallel if they do not intersect and two planes in space are parallel if they do not intersect. If two distinct planes intersect then their intersection is a line and if two distince lines intersect then their intersection is a point. Discuss skew lines and intersection and parallelism between lines, planes, segments, and rays.

E. Parallelism (cont.)

References
S: Chapter 2
R: 2.1
K: 3.4

3. Measurement of line segments and angles.

Let the students determine a unit for linear measure and discuss measure of segments, considering different "sizes" of unit measure. Then discuss and convert English measures so that they will become acquainted with meters. A unit of measure for angles can then be considered and measure of angles. Discuss the relationships between measure of angles and segments.

References
S: Chapter 3
R: 2.2, 2.3

4. Simple Closed Curves, Polygons, and Polyhedra

A. Simple Closed Curves

Define a simple closed curve as a closed curve which does not intersect itself. Then a polygon is a simple closed curve which is the union of a finite number of straight line segments. Briefly consider the topic of convex sets. Mention the names of some special polygons and the difference between regular and non-regular polygons. A polygonal region can be defined as the union of a polygon and its interior, and a polyhedron as a closed surface consisting entirely of polygonal regions. Discuss some of the regular polyhedra and how to construct them. Actual models would be nice to used for demonstration.

References
S: Chapter 4
R: 2.1, 5.1
K: p. 43-45

B. Quaderilaterals

Two lines are perpendicular if they intersect in such a way that the four angles formed are congruent. Discuss perpendicular lines and planes. Make the following definitions: a trapezoid is a quadrilateral with at least one pair of parallel sides, a parallelogram is a quadrilateral with opposite sides parallel,

B. Quaderilaterals (cont.)

a rectangle is a parallelogram with four congruent angles, a rhombus is a parallelogram with four congruent sides, and a square is a rectangle with four congruent sides.

References
S: Chapter 4
R: 2.2, 3.1, 3.2, 3.5, 5.3

C. Parallelism

Discuss the parallel axiom. Briefly introduce the concept of a transversal and derive the properties of the angles associated with two parallel lines and a transversal. Prove ~~the exterior angle theorem and~~ discuss parallelograms and their properties.

References
R: 3.1 - 3.5

D. Additional Topics

Prove that the sum of the measures of the angles of any triangle is 180°. Note proof in outline. Prove that the midpoints of the sides of any quadrilateral determine a parallelogram.

References
K: Theorem 6.7

5. Circles

Introduce and discuss some of the properties of circles, e.g. chords, central angles, and arcs. Omit the section on circumference and select some theorems from the following section.

References
S: Chapter 5

6. Geometric Constructions

Give the basic constructions. Show that the trisection of any angle (other than particular examples) is impossible.

References
S: Chapter 6
R: 12.1 - 12.5

7. Congruence

A. Triangle Congruence

Develop the definition of congruence for triangles. Take SAS as an axiom and then prove the ASA and SSS are sufficient conditions for triangle congruence. The ASA theorem can be proved directly, but it is necessary to introduce the isosceles triangle and prove that the base angles are congruent before proving SSS. Note that SAA is also a sufficient condition for congruence but do not prove it.

B. General Concept of Congruence

Redefine congruence as a one-to-one distance preserving correspondence between sets of points. Give examples of translations, rotations, and reflections and show that congruence of triangles is preserved under these transformations. Develop the same ideas for other plane figures. Show that congruence is an equibalance relation and give some examples of equivalence classes. Then use the unit from Math Teacher to lead into a discussion of similarity.

References
S: Chapter 7
R: 2.3, 2.4, p. 32
K: 4.4
Math Teacher, April 1968

C. Similarity

Discuss symmetry of figures with respect to a point or line and give several examples of symmetric figures. Briefly discuss symmetry of space figures on your own. Two figures are said to be similar if they have the same shape but not necessarily the same size. Discuss similarity for polygons, polyhedra, circles, and spheres.

References
S: p. 94
R: 2.2 - 2.3

8. Area of Plane Figures

Use a square unit as a basis for area. Show that the area of a rectangle can be found as a • b where a and b are the lengths of two adjacent sides of the rectangle and may be any positive real numbers. Standard units should be stressed. Determine the area of triangles, parallelograms, trapezoids, and convex polygons. A triangulation process is convenient for the latter. Consider the measurement of circumference, area, and arc length in circles. An experimental verification that π is a constant is a good exercise.

References
S: Chapter 8
R: 4.1 - 4.7

9. Errors in Measurement

Consider briefly approximate measurement, relative error, indirect measurement, and applications.

References
S: Chapter 9
R: 2.8

10. Similarity

Before giving a rigorous definition of similarity, discuss scale drawings which is the basis for studying similarity. Give some sufficient conditions for similarity of triangles, e.g. SAS, SSS, AAA. Discuss similarity for other plane figures and space figures and application of indirect measure.

References
S: Chapter 10
R: 2.6, 2.7

11. Algebraic Representation of Geometric Figures

Introduce the concept of coordinatization of the line and plane. The graph sets of ordered pairs in the plane. After developing the equation of a straight line, discuss linear inequalities in the plane.

References
S: Chapter 11 (omit trig. ratios)
M: Chapter 8

12. Geometry of Selected Space Figures

A. Theorems on Lines and Planes

Show how a plane is determined. Discuss parallels and perpendiculars to planes.

References
S: Chapter 12
R: 5.2 - 5.4

B. Dihedral and Polyhedral Angles

Provide illustrated definitions of a halfplane, dihedral angle, plane angle, measure of a dihedral angle, polyhedral angle, face angle, and congruence of the above. Numerous examples from the physical world should be given.

References
S: Chapter 12
R: 5.5

C. Polyhedra

State Euler's Formula and give a number of examples. Also prisms are defined. Develop the analogy of simple closed surfaces to simple closed curves in a plane. A great circle is defined.

References
S: Chapter 12
R: 6.1 - 6.4
K: 4.12

13. Locating points in space

Using the concept of great circle, discuss briefly longitude and latitude. Omit coordinates for points in space.

References
S: Chapter 13 (omit coordinates for points in space)

14. Volume and Surface Area

Define the surface area of a polyhedron as the sum of the areas of its faces. Use a cubic unit as a basis for volume. Provide some justification for the formulas given for the volumes of polyhedra and cones. Discuss

14. Volume and Surface Area (cont.)

surface area and volume of a sphere. The instructor should try to justify the formulas given in this section in an informal way and it is not desirable to provide proofs. Do not require excessive memorization of formulas.

References
S; Chapter 14
R: 6.5 - 6.13

15. Formal, Informal, and Experimental Geometry

A formal development proceeding from undefined terms through axioms to theorems is not the primary intent of this course. However, a discussion is in order which compares elements of informal geometry and problems which may be solved by observation or trial and error to a body of knowledge which results from an organized logical development of geometry. The former deals with certain properties of objects which can be seen and touched in the physical world. The latter involves properties of objects which can only be imagined and it utilized deductive reasoning to derive relationships between postulates and theorems. From this framework we can develop the nature of proof in geometry and how this leads to other geometries besides Euclidean geometry.

References
S: Chapter 15
R: 1.3
K: Chapter 1

16. Modern Geometry

A. Topology

The student need only to become acquainted with some of the terms and concepts contained in Topology.

B. Finite Geometry

Omit Projective Geometry. The students should see a completely axiomatic system. Thus for introduction into finite geometry, state axioms, definitions, and theorems.

References
M: 10.5

16. Modern Geometry (cont.)

C. Non-Euclidean Geometry

Discuss the evolution of geometry from ancient times to the present thus including hyperbolic and elliptic geometry. Consider Euclid's contributions to the subject in particular his fifth postulate. Thus the alternatives to Euclid's fifth postulate lead to two different non-Euclidean geometries. If time permits consider some of the properties of projective geometry for prospective. For more discussion on non-Euclidean Geometry see Measure and Sobel.

References
S: Chapter 16
M: 10.1 - 10.3

Grading Standards for T106

For insuring standard grades in T106, the following should be mastered for a grade of "C".

1. Be able to use protractor to measure an angle.
2. Be able to define angle, acute angle, right angle, obtuse angle, triangle, acute triangle, right triangle, obtuse triangle, median of a triangle, exterior angle, quadrilateral, parallelogram, rectangle, rhombus, and square.
3. Find perimeter of any polygon, given lengths of sides.
4. Find circumference of a circle.
5. Construct line segment, midpoint of a segment, perpendicular to a line at a given point, bisector of an angle; copy given angles and triangles (all with straightedge and compass).
6. Determine if two triangles are congruent or similar.
7. Recognize symmetry.
8. Find area of a triangle, rectangle and square.
9. Make scale drawings of plane figures.
10. Define surface area and volume of polyhedrons, cylindrical figures, cones, pyramids, and spheres.
11. Be able to use Pythagorean Theorem.

III. BIBLIOGRAPHY

A Adler, C. F., *Modern Geometry: An Integrated First Course*. McGraw-Hill Book Company, Inc., New York, 1958.

B Brumfiel, C. F., Eicholz, R. E., and Shanks, M. E., *Geometry*. Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1960.

C Coxeter, H. S. M., *Introduction to Geometry*. John Wiley and Sons, Inc., New York, 1961.

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R Ringenberg, L. A., *Informal Geometry*. John Wiley and Sons, Inc., New York, 1967.

S Smart, J. R., *Introductory Geometry: An Informal Approach*. Brooks/Cole Publishing Company, Belmont, California, 1967.

Y National Council of Teachers of Mathematics, *Enrichment Mathematics for the Grades, Twenty-Seventh Yearbook*. The National Council of Teachers of Mathematics, Washington, D. C., 1963.

SPECIAL MATERIALS ARE ON RESERVE IN SWAIN WEST LIBRARY.

Explorations that add to pupils' understanding of geometric facts prior to geometric proofs

Congruence Geometry FOR JUNIOR HIGH SCHOOL*

By WALTER J. SANDERS
and J. RICHARD DENNIS

University of Illinois
Urbana, Illinois

BECAUSE Euclid did such a superb job in organizing the geometry of the plane, the practice has been to follow his example almost exclusively in preparing elementary and high school materials. The results at the elementary and junior high levels have been limited to a presentation of point-set definitions, with at best a minimal exploration of the geometric properties of the figures defined. This reluctance to include the study of relationships between figures may be a consequence of difficulties inherent in the standard organization.

Several mathematicians have presented postulational organizations of geometry based on the isometries of the plane.[1] While none of these formal developments are suitable for junior high students, they have shown the *mathematical* feasibility of such an approach. To make this approach *pedagogically* feasible, the ideas must be presented in such a way as to be readily understood by youngsters. The presentation should also give each pupil a foundation on which to base conjectures concerning possible relationships and, eventually, to verify assertions.

In this paper we shall show an approach to plane geometry based on isometries which is suitable for junior high youngsters. First, tracings are used to establish a notion of congruence. Second, three types of motions—slides, turns, and flips—are described. With just these motions, a tracing can be moved from a position of coincidence with one figure to a position of coincidence with any other congruent figure. Third, the tracing motions are used to define and study the three basic isometries—translations, rotations, and reflections. Finally, properties of these isometries are used to establish the conventional congruence geometry properties.

Congruence

Intuitively, two geometric figures are *congruent* if they are the same size and shape. One way to show that two figures are congruent is to make one of them fit exactly on the other, that is, to use superposition. So, if the figures are drawn on paper and it is possible to separate the figures by cutting the paper, make such a cut and then place one figure over the other to see if they fit.

The triangles in Figure 1 are congruent. But, since they overlap, it is not possible to separate them by cutting the paper. The triangles can be shown to be congruent,

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* The ideas presented in this paper arose out of the authors' work with UICSM in developing materials for culturally disadvantaged junior high school students under a grant from the National Science Foundation.

[1] An isometry is a distance-preserving function of the space. See, for example, *L'enseignement de la géométrie* by G. Choquet (Paris: Hermann).

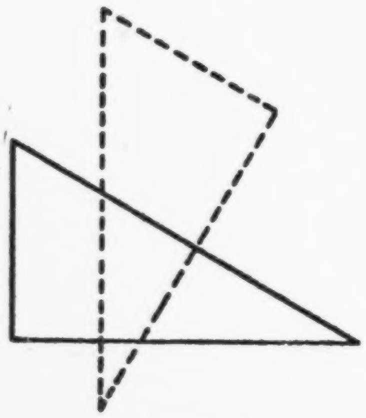


FIGURE 1

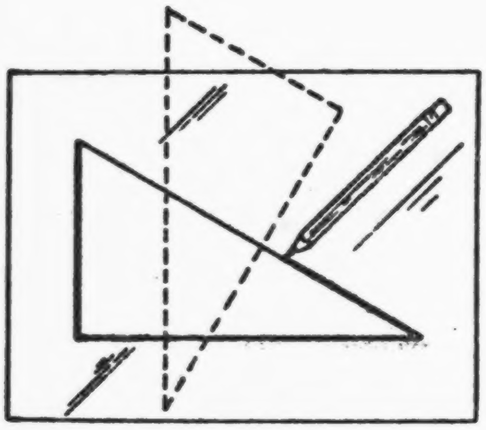
however, by making a tracing of one of the triangles and using it to do the matching. Figure 2 shows this process: first to trace one triangle, then to match the tracing with the other triangle.

Children become aware of the basic properties of congruence through considerable practice in using tracings to compare figures. To be congruent, figures must be alike in size and shape. Figures which differ in at least one detail of size or shape are not congruent. (The tracing test shows this within the accuracy of the tracings.)

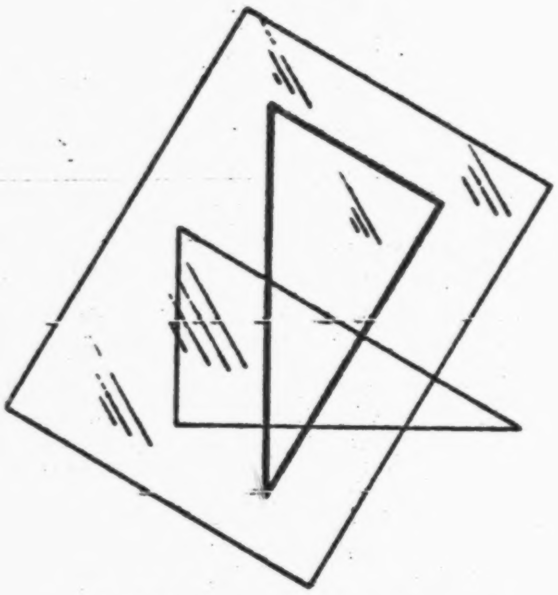
For each property of comparison by tracings there is a corresponding property of congruence. Table 1 shows this correspondence for three familiar properties of congruence.

Relations that are reflexive, symmetric, and transitive are called *equivalence relations.* Since congruence has these properties, congruence is an equivalence relation.

Another fundamental property of congruence that can be made clear through the use of tracings is that corresponding



(a) Trace one triangle



(b) Match the tracing with the other triangle.

FIGURE 2

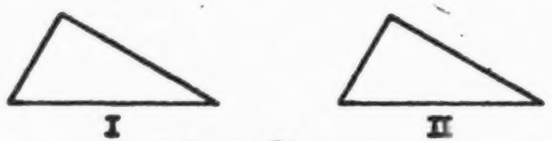
parts of congruent figures are congruent. Figure 3(a) shows a pair of congruent triangles. The altitude to the base of triangle I must be congruent to the corresponding altitude of triangle II, since a

TABLE 1

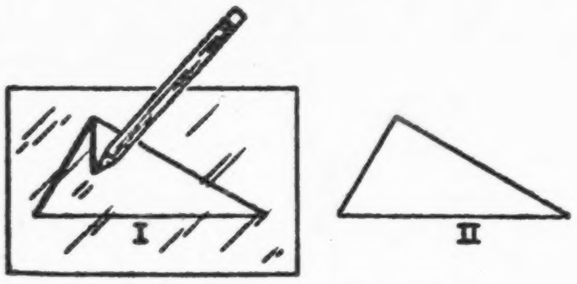
COMPARISON OF TRACING PROPERTIES WITH CONGRUENCE PROPERTIES

Tracing Property	Congruence Property
1. A tracing of a figure matches that figure.	1. A figure is congruent to itself (reflexivity).
2. If a tracing of a first figure matches a second figure, then a tracing of the second will match the first.	2. If a first figure is congruent to a second, then the second is congruent to the first (symmetry).
3. If a tracing of a first figure matches a second and a tracing of the second matches a third, then a tracing of the first will match the third.	3. If a first figure is congruent to a second and the second is congruent to a third, then the first is congruent to the third (transitivity).

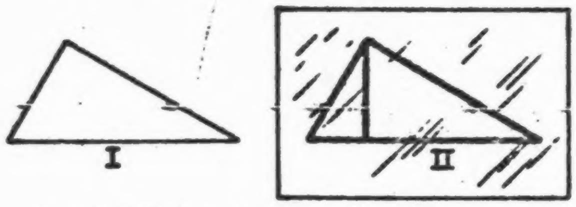
tracing of triangle I that includes this altitude *must* match triangle II. See Figures 3(b) and (c).



(a) $\triangle I \cong \triangle II$



(b) Trace $\triangle I$ with altitude



(c) Match tracing with $\triangle II$

FIGURE 3

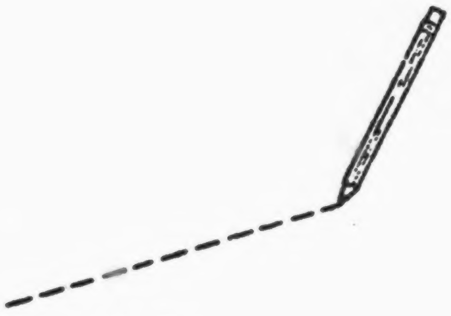
Three Basic Motions

A piston in a gas engine *slides* up and down; shadows slide along the ground; a gear in a clock *turns* on a shaft; stars appear to turn around the North Star. Sliding and turning motions occur again and again in the world about us. Many applications of mathematics, including many of a geometric nature, are concerned with these two motions.

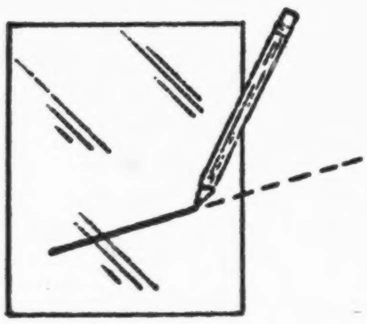
A sliding motion is a motion along a straight line without any accompanying twisting or turning. A sliding motion can be made with a sheet of paper by using a guideline as shown in Figure 4.

A turning motion is a motion along a circle with the center of the circle fixed (keeping its position). To make a turning motion with a sheet of paper, hold one point fixed by using something pointed, such as a pencil, as shown in Figure 5.

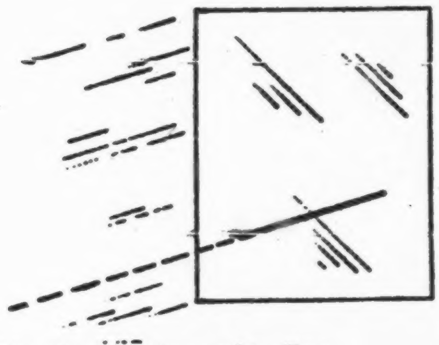
A *slide* is a sliding motion of specified distance and direction. Both distance and direction can be indicated conveniently with an arrow. Figure 6 shows how to slide a sheet of paper as indicated by an arrow.



(a) Draw guide line



(b) Trace guide line

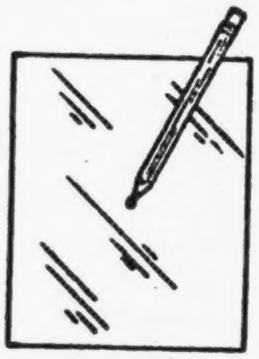


(c) Slide sheet using guide line

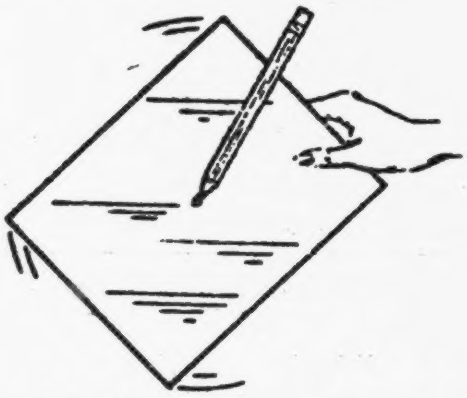
FIGURE 4

A *turn* is a turning motion of specified amount and direction. Amount and direction of turn and location of turn center can be given by a curved arrow and a dot. Figure 7 shows how to turn a sheet of paper as indicated by a turn arrow and dot.

In Figure 8, one figure is drawn with a pattern as in (a); then, without lifting the pattern from the page, it is moved to a new position and used to draw a second figure as shown in (b). The two figures, shown in (c), must be congruent, since they were



(a) Hold down with pencil point



(b) Turn sheet, keeping point stationary

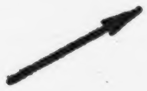
FIGURE 5

drawn from the same pattern. The pattern is like a tracing in this case.

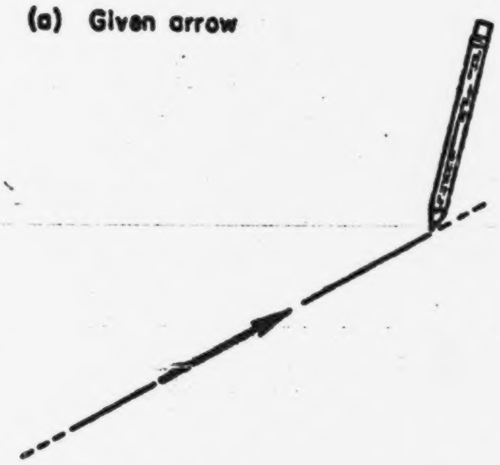
Figure 9 shows how a tracing can be moved from a position of coincidence with one of the figures of Figure 8 to a position of coincidence with the other by first making a slide, then a turn.[2]

The tracing can also be moved from a position matching the first figure to one matching the second figure with just a *single turn,* without sliding. Figure 10 shows how a turn about a point between the two figures will do this.

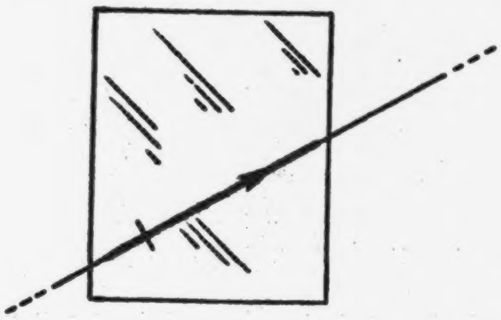
This raises a general question: Given two congruent plane figures, can a tracing of one of them be moved by just a slide or by just a turn so that the tracing exactly matches the other figure? To answer this question, consider the two figures in Figure 11. In this case, after one of the figures was made with a pattern, the pattern was flipped over, so that the other side faced up. Then the second figure was drawn. A little experimentation moving a tracing around appropriately is all that is needed to convince one that a tracing which is lined up with one of the figures cannot be



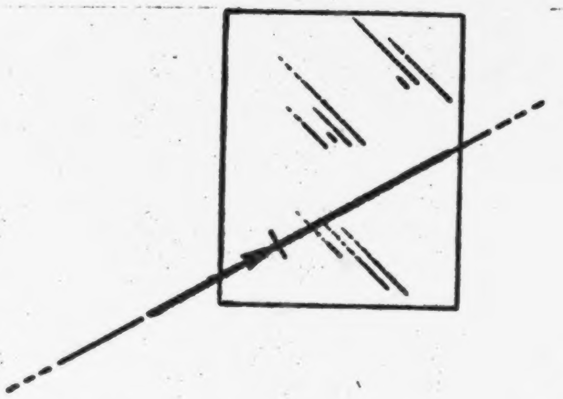
(a) Given arrow



(b) Draw guide line



(c) Trace guide line and mark tail of arrow



(d) Slide tracing until mark is at tip of arrow

FIGURE 6

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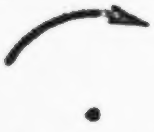
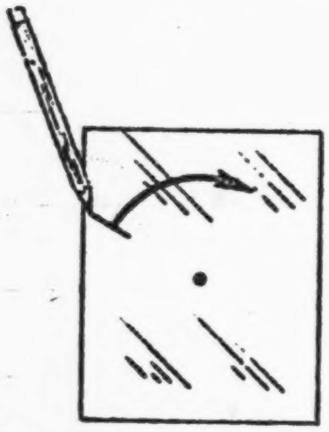
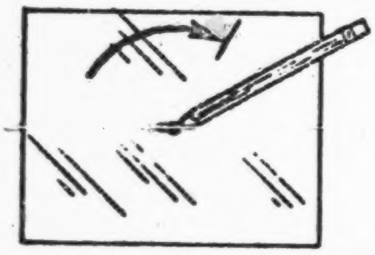
[2] There are, of course, many other combinations of slides and turns that will move the tracing from coincidence with one figure to coincidence with the other.



(a) Given turn arrow and turn center



(b) Mark tail of arrow



(c) Fix turn center and turn until mark is at tip of arrow

FIGURE 7

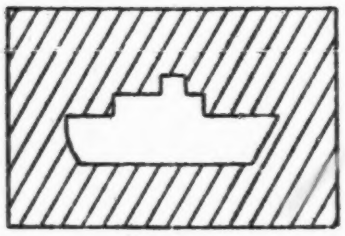
moved to a position matching the other figure unless the tracing is flipped over. Since slides and turns do *not* reverse sides of the tracing, no combination of slides and turns can be found that will move a tracing from a position matching one of the figures to a position matching the other.[3]

If you place a mirror appropriately between the two figures of Figure 11, as shown in Figure 12, and look into the mirror, what you see in the mirror will be the other figure. For this reason, we call the figures *mirror images.*

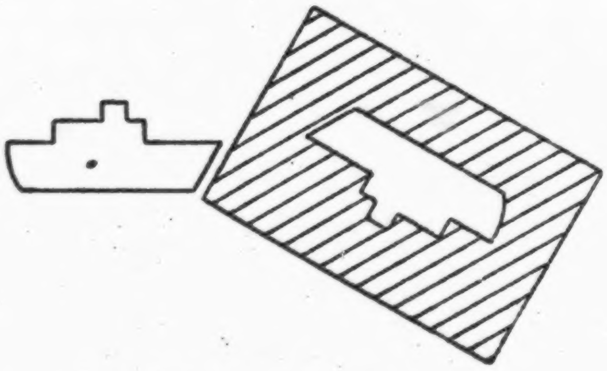
If you now mark dots on the paper at each end of the mirror and draw a line through the dots, a fold along the line will match the two figures, as shown in Figure 13. We can use this "fold line" to describe a motion which will move a tracing of one of the figures to a position matching the other. This motion, which we will call a *flip,* is shown in Figure 14, in which a tracing of one figure is flipped to match a second figure. The "line of fold" will be called the *flip line.*

WE CAN now finish the discussion of types of motions which are sufficient for moving a tracing of one plane figure to coincide with a second congruent figure. It is always possible to accomplish such a move by performing one of the following motions:

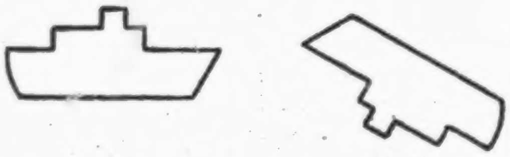
1. a single slide
2. a single turn



(a) Draw first figure



(b) Draw second figure



(c) Resulting congruent figures

FIGURE 8

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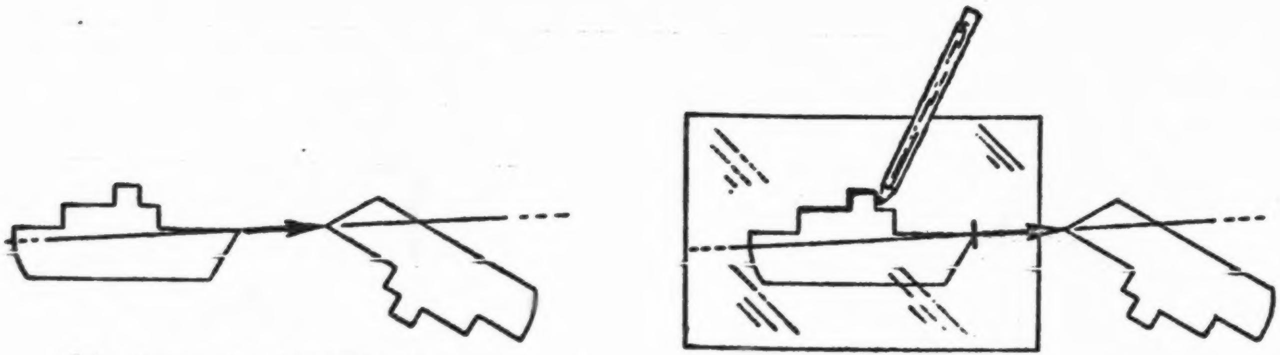
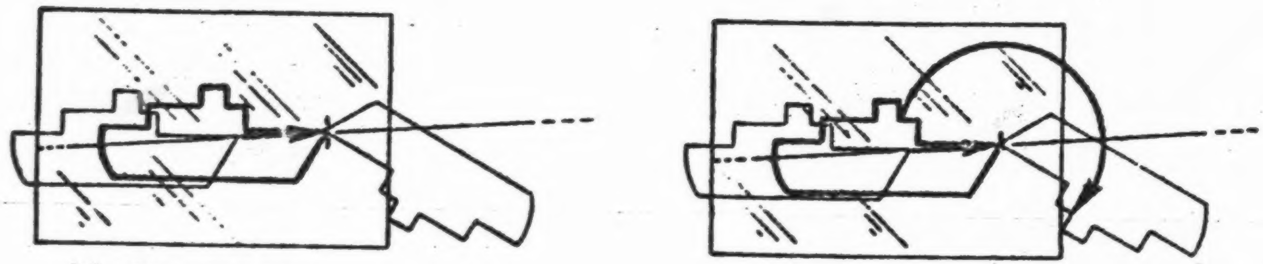
[3] Although one cannot, in general, use just slides and turns to move a tracing of a figure to coincide with another congruent figure, it is the case that whenever a combination of slides and turns will move a tracing from one position to another, there is either a single slide or a single turn that will accomplish the change in position. (This result is not beyond the reach of junior high students.)



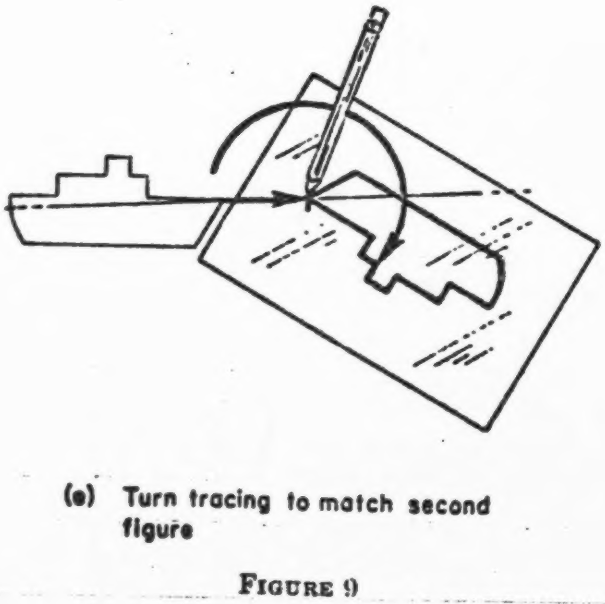
(a) Arrow showing slide to make

(b) Trace first figure, guide line, and tail of arrow



(c) Slide tracing

(d) Arrow showing turn to make



(e) Turn tracing to match second figure

FIGURE 9

3. a single flip followed by a single slide
4. a single flip followed by a single turn

Of course, there are many other combinations of slides, turns, and flips which will also work. The more interesting of these are concerned with flips: a tracing of any plane figure can be moved to a position of coincidence with any other congruent figure by one flip, or by two flips, or by three flips—no other motion is necessary. To see this, one need only convince himself that any slide can be accomplished by two successive flips, and any turn can also be accomplished by two successive flips.[4]

Isometries

Translations, rotations, and reflections

The slide indicated in Figure 15(a) will move a tracing sheet so that a tracing of

---

[4] The authors were pleased to discover that at least one high school geometry text suggests imagining a combination of slide, turn, and flip motions to carry out the standard superposition arguments in triangle congruence theorems. See Barnes and Hendrix, *Plane Geometry and Its Reasoning* (New York: Harcourt, Brace & Co., 1937).

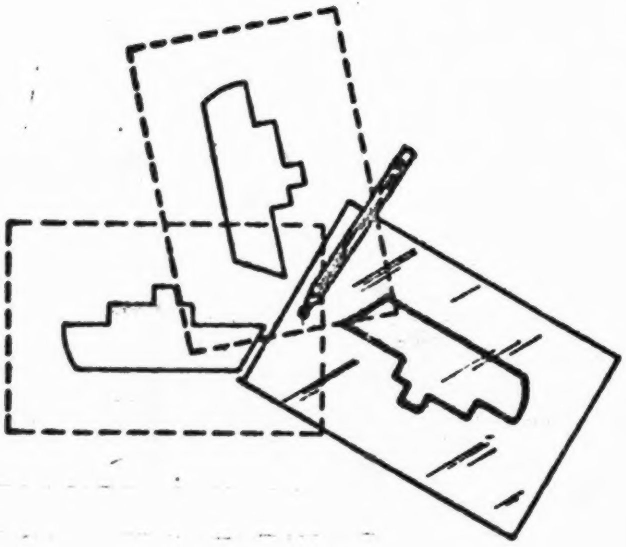


FIGURE 10

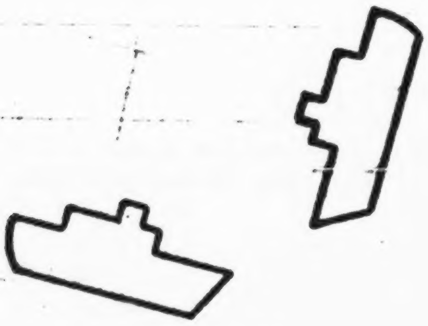


FIGURE 11

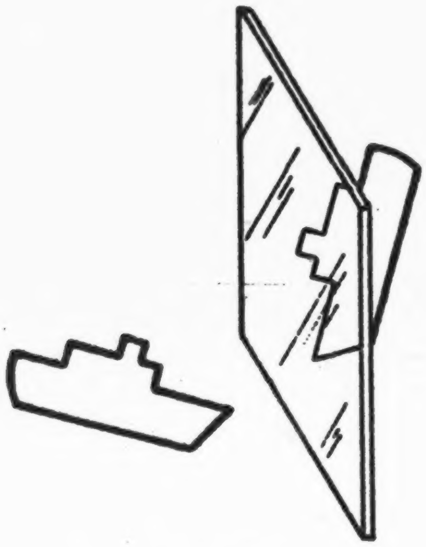
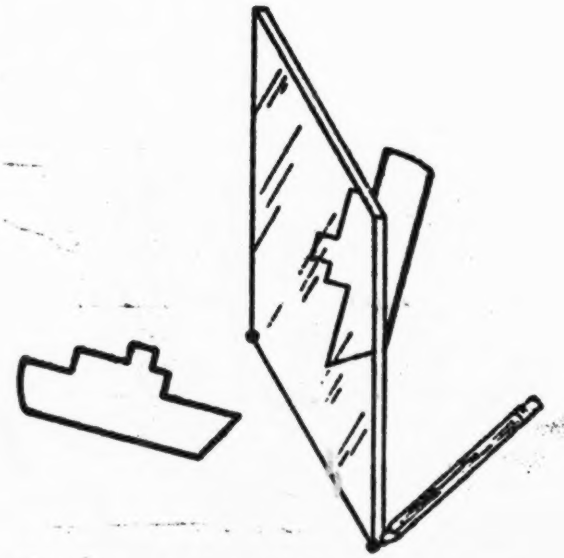
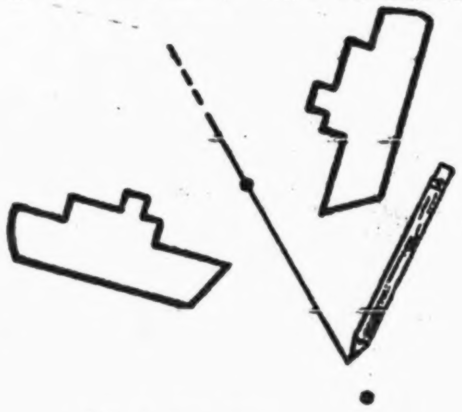


FIGURE 12

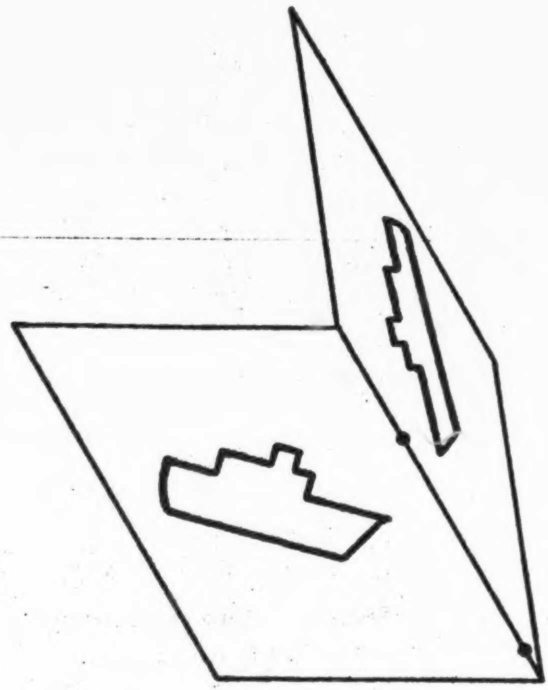
$\triangle ABC$ matches $\triangle XYZ$. As the slide is made, the tracing of vertex A moves from A to X, as shown by the guideline in Figure 15(b). Similarly, the tracings of vertices B and C move from B to Y and from C to Z, respectively. The *map arrows* in Figure 15(c) show the pairing of the points



(a) Mark dots at ends of mirror

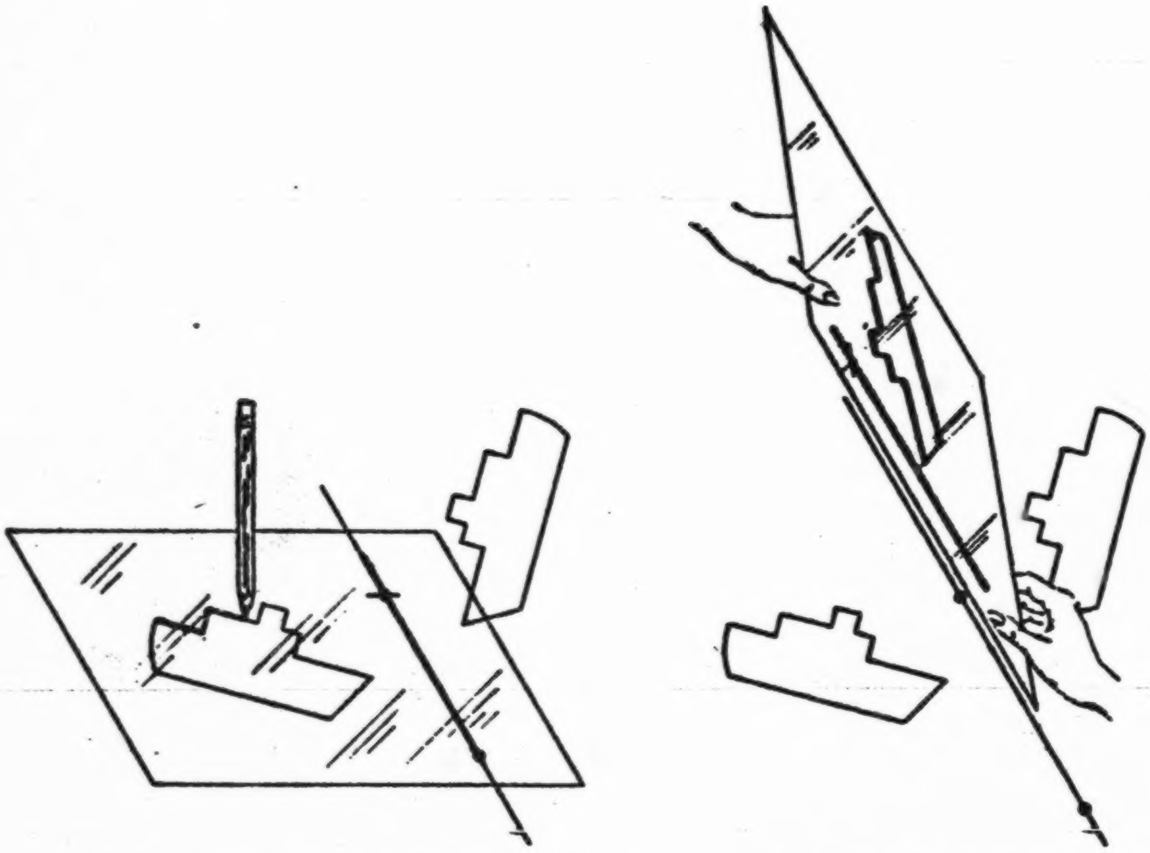


(b) Draw line



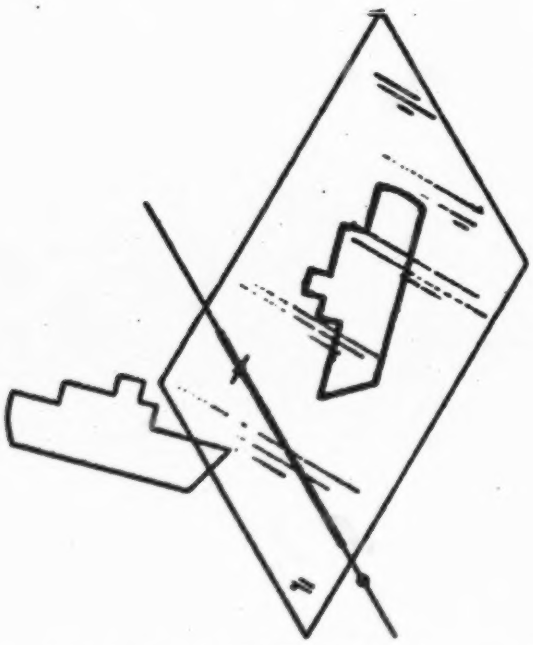
(c) Fold along line

FIGURE 13



(a) Trace one figure and the flip line, and mark a reference point on the flip line

(b) Hold tracing at ends of flip line and flip

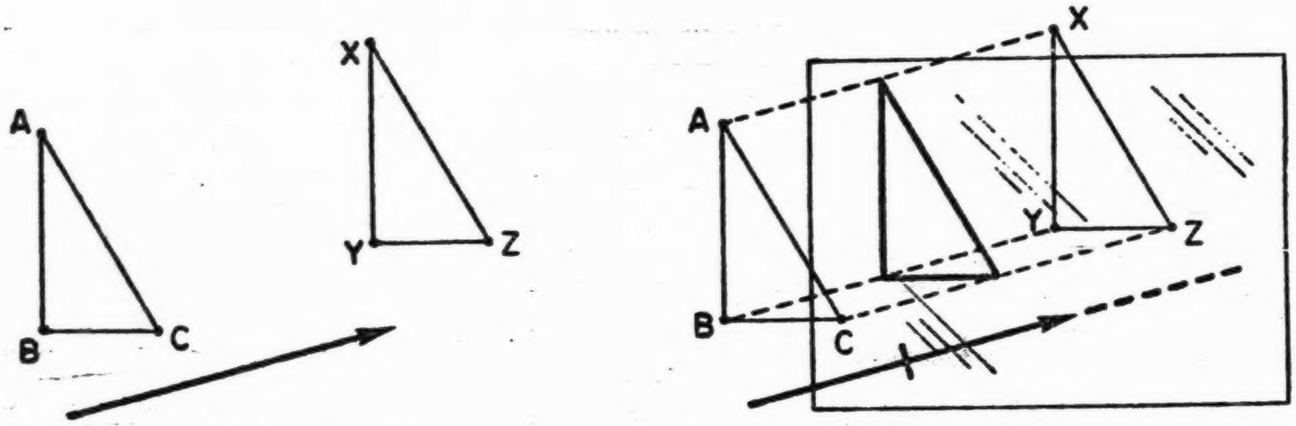


(c) Line up flip line and reference dot. When finished, the tracing will match the second figure.

FIGURE 14

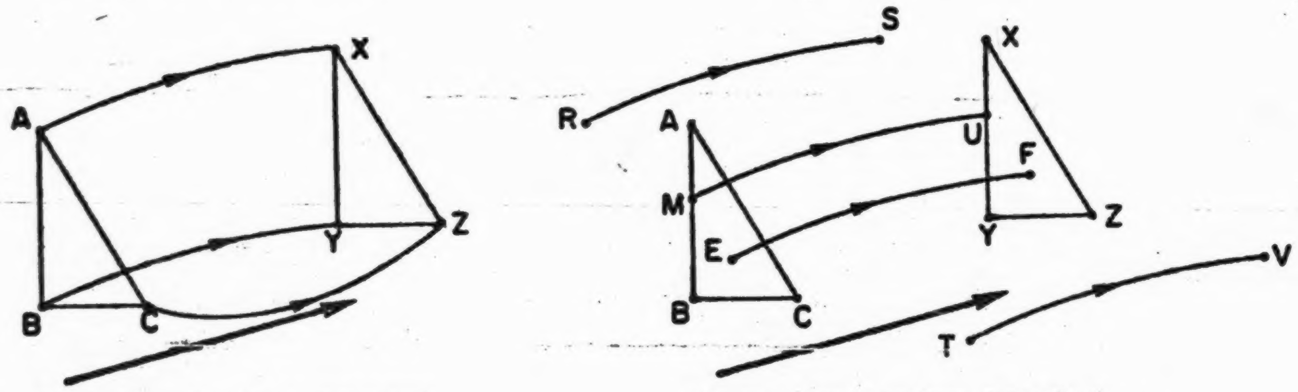
A, B, and C with points X, Y, and Z.

Sliding a tracing in this manner shows one way of pairing each point of $\triangle ABC$ with a unique point of $\triangle XYZ$. It also



(a) Given ΔABC, ΔXYZ, slide

(b) Trace ΔABC and slide



(c) Pairing of vertices determined by the slide

(d) Some other pairings determined by the slide

Figure 15

shows how to pair other points. To locate the point which pairs with any point P, see where a tracing of P is moved to by the slide. The map arrows in Figure 15(d) show several such pairings. Point M, the midpoint of side AB of $\triangle ABC$, pairs with the midpoint U of side XY of $\triangle XYZ$. Point E, an interior point of $\triangle ABC$, pairs with point F, which is interior to $\triangle XYZ$.

The set of all ordered pairs of points determined by a slide is a one-to-one function called a *translation*. For each ordered pair, the first component is called the *original*, and the second component is called the *image* of that original. For the translation shown in Figure 15, we see that X is the image of A, U is the image of M, and S is the image of R. Also, the set of images of the points of a given figure is called the *image* of that figure. So, for the translation shown in Figure 15, $\triangle XYZ$ is the image of $\triangle ABC$, and the image of segment AB is segment XY.

The one-to-one function determined in a similar way by a turn about a point is called a *rotation*. Figure 16 shows a rotation about point T. Under that rotation $\triangle XYZ$ is the image of $\triangle ABC$. The map arrow shows that point U is the image of D, while the "loop" map arrow shows that point T is its own image.

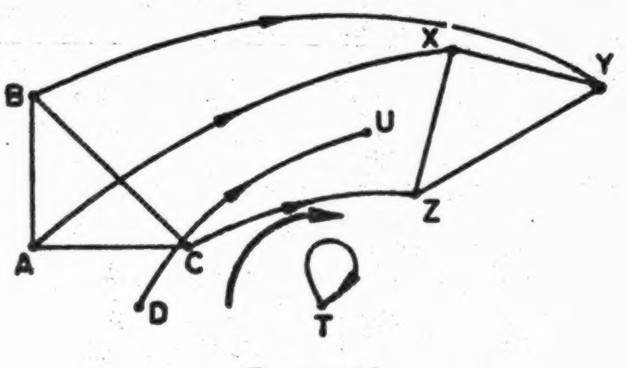


Figure 16

Each one-to-one function determined by a flip about a line is called a *line reflection*. The map arrows in Figure 17 show some of the original-image pairs for the reflection about line *l*. Under a flip about line *l* a tracing of $\triangle ABC$ will be moved to coincide with $\triangle XYZ$. So $\triangle XYZ$ is the image of $\triangle ABC$ under the reflection about *l*.

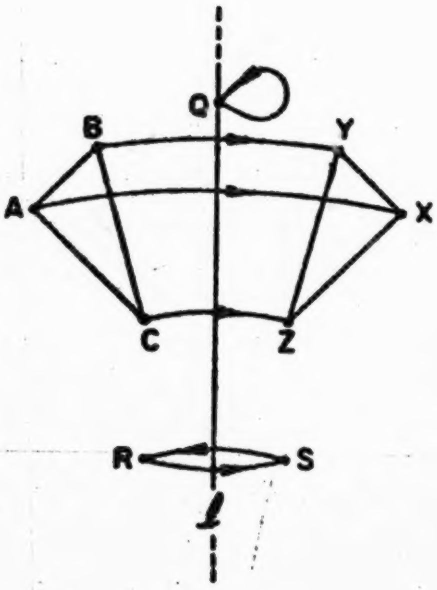


FIGURE 17

The map arrow from *R* to *S* shows *S* to be the image of *R*, while the map arrow from *S* to *R* shows *R* to be the image of *S*. That is, *R* and *S* are images of each other. The loop arrow at *Q* shows that *Q* is its own image. Because of the way flips work, we see that if *M* is the image of *N*, then *N* is also the image of *M*. So $\triangle ABC$ is the image of $\triangle XYZ$. Also, each point of line *l* is its own image.

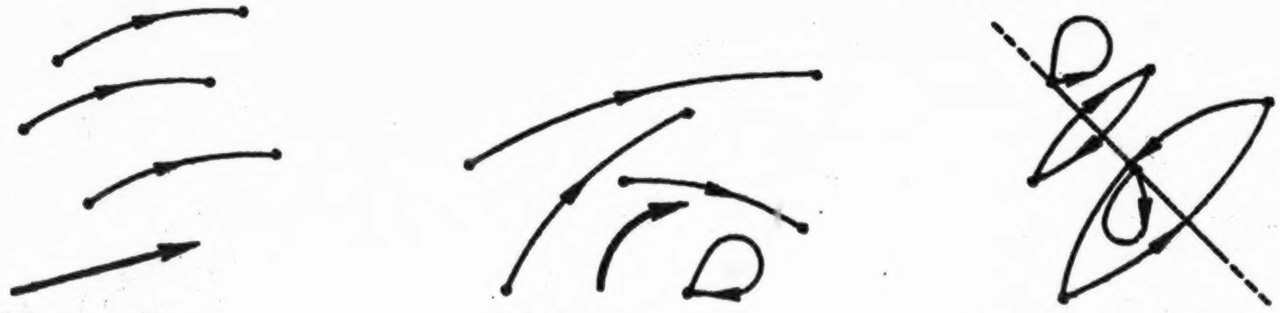
For each translation, rotation, and line reflection, any figure will be congruent to its image. This follows from the way tracings were used to define these functions. In particular, the image of a segment will be a congruent segment. Stated another way: The distance between any two points is the same as the distance between their image points. For this reason, translations, rotations, and line reflections are called *distance-preserving functions.* Distance-preserving functions of the plane are sometimes called *isometries* of the plane; so translations, rotations, and line reflections are isometries of the plane.

Fixed points

A point which is its own image (with respect to a given function) is referred to as a *fixed point.* A map arrow for a fixed point starts and ends at that point. Figure 18 shows several pairings for a translation, a rotation, and a reflection.

There are no fixed points for the translation in Figure 18(a). To see this, think of sliding a tracing as indicated by the slide arrow; every part of the tracing would slide the same distance and direction. Therefore the distance between a point and its image is constant; it is the same as the length of the slide arrow. Finally, if a translation has at least one fixed point, *all* points must be fixed, and the translation is the identity function.

The only fixed point of the rotation in Figure 18(b) is the turn center. In fact, the turn center is always a fixed point for a rotation. However, there may be other fixed points, as in a 360-degree rotation. But, if there are two fixed points, then *all* points are fixed and the rotation is the identity function. (These results are most easily seen by thinking of tracings. As a tracing is turned, the turn center is held



(a) Translation (b) Rotation (c) Line reflection

FIGURE 18

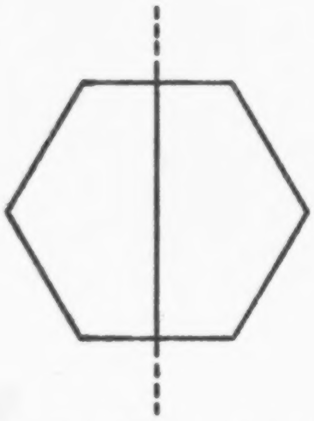
fixed. If a second point of the tracing turns back to where it started, the entire tracing will be back to where it started.)

By considering a flip of a tracing sheet about a line, one can see that the fixed points for a line reflection are exactly the points of the flip line.

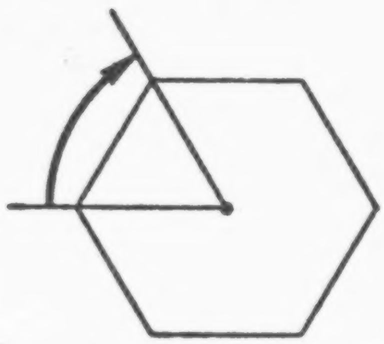
Invariance

A figure which is its own image under a function is said to be *invariant* under the given function. Under the reflection about the line shown, as in Figure 19(a), the image of the regular hexagon is the hexagon itself. In Figure 19(b) the hexagon is invariant with respect to a 60-degree rotation about its center. The only translation which leaves a hexagon invariant is the (trivial) identity translation.

The distinction between fixed points ~~and invariance can be seen by referring~~ to Figures 19(a) and 19(b). In both illustrations, the hexagon is invariant. For the line reflection the hexagon has exactly two fixed points. These are the two points where the hexagon crosses the flip line. For the rotation, the hexagon has no fixed points.



(a) Invariance under a line reflection



(b) Invariance under a rotation

FIGURE 19

Now consider the translation shown in Figure 20. A tracing of line *n* "slides along" line *n*. This means that the corresponding translation pairs each point of line *n* with a point of line *n*, i.e., the line *n* is *invariant* (mapped onto itself) under this translation. Notice, however, that no point of line *n* is fixed.

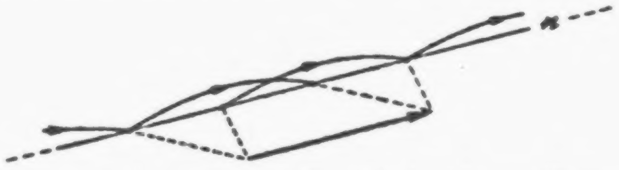


FIGURE 20

Geometry

We are now in a position to tackle the conventional theorems of plane geometry. Our tools will be properties of translations, rotations, and reflections. In the following sections we offer some samples of this analysis.

Parallel lines

After examining the effect of translations upon various lines in the plane, students formulate the following generalization:

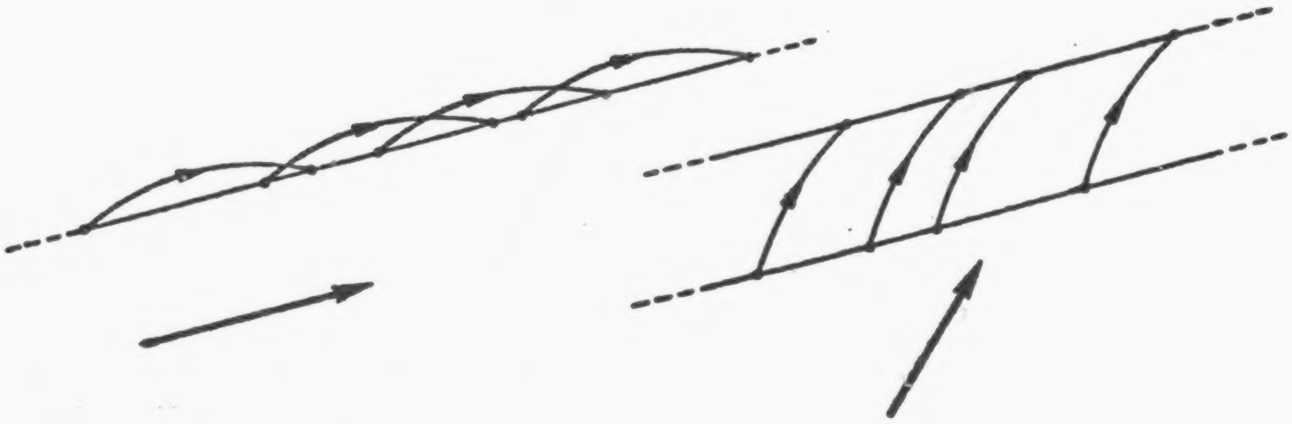
For each line in the plane and for each translation, either the translation maps the line onto itself, or else the line and its image have no points in common (Fig. 21).

With this observation, we are in a position to define "parallel lines." We say that lines are parallel whenever there is a translation that maps one line onto the other. With this definition a line is parallel to itself.

Vertical angles

An important property of all 180-degree rotations is that they map any line onto a parallel line and, specifically, map any line through the center of rotation onto itself. See Figures 22(a) and 22(b).

This property plays a central role in the



Line mapped onto itself

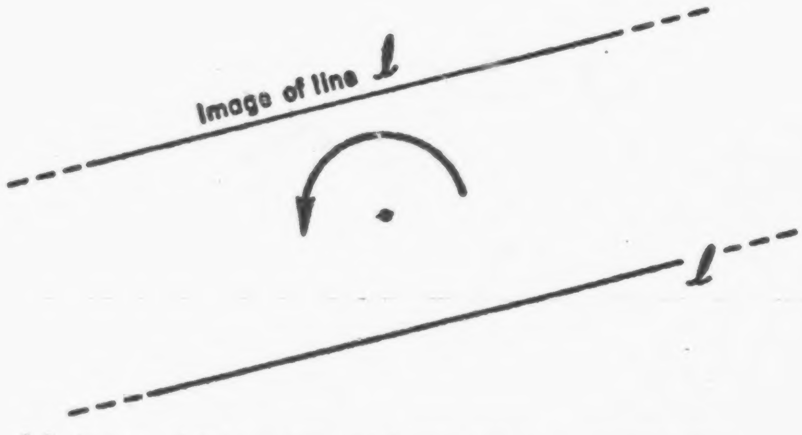
No points common to line and image

FIGURE 21

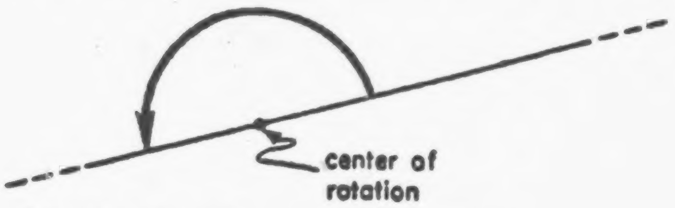
study of vertical angles. Two lines that intersect at a point P (see Fig. 23) form four angles, opposite pairs of which are called *vertical angles.* A 180-degree rotation about point P maps each of the lines onto itself. In particular, it maps ray PA onto ray PD and ray PB onto ray PC; i.e., it maps $\angle APB$ onto $\angle DPC$. Therefore, these vertical angles are congruent. This same rotation also maps $\angle APC$ onto $\angle DPB$, so these vertical angles are also congruent.

Parallel lines–transversal theorems

Given a pair of parallel lines, the translation which maps any point of the first line onto some point of the second line maps all of the first line onto the second line. Also, each translation preserves the order of points along a line, and each translation maps a set onto a congruent set. These properties give us tools for exploring the standard parallel lines–transversal theorems.



(a) Image of line under 180° rotation is a parallel line



(b) Line through turn center invariant under 180° rotation

FIGURE 22

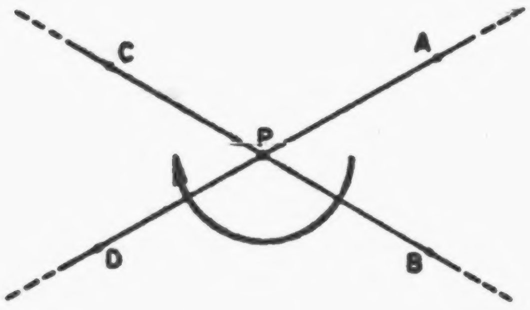


Figure 23

Suppose that line *AC* is parallel to line *BD* (Fig. 24), and that line *n* crosses each as shown. Consider the translation that maps *A* to *B*. This translation maps line *AC* onto line *BD*. In particular, this translation maps ray *AC* onto ray *BD* and ray *AB* onto ray *BE*; i.e., it maps $\angle CAB$ onto $\angle DBE$. Therefore, $\angle CAB$ is congruent to $\angle DBE$. This same translation establishes the congruence of the other pairs of corresponding angles.

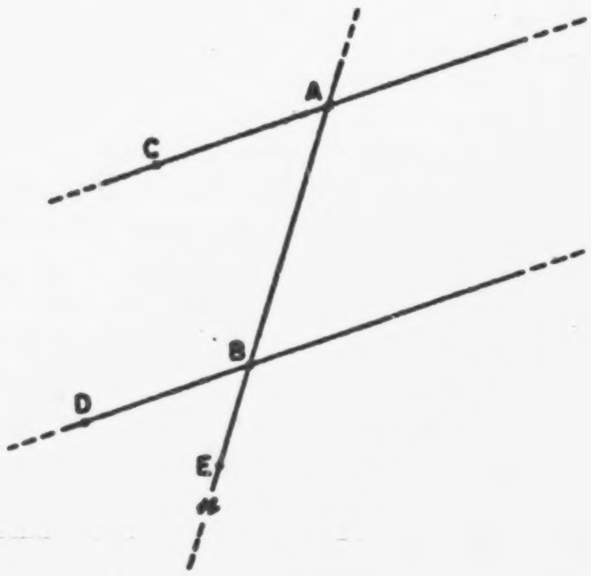


Figure 24

At first it is best to illustrate arguments like this with the use of tracings. Students who trace $\angle CAB$ and slide the tracing from *A* to *B* readily see that the tracing then matches $\angle DBE$. Although initial arguments of this type should be discussed with the help of tracings, eventually students should be encouraged to make their arguments on a more verbal basis, with a minimum of direct help from the tracings.

Perpendicular lines

Invariance plays an important role in the development of geometric ideas. Some lines are invariant under reflection about another line, and some lines are not. In the case where a line is invariant under reflection about another line (Fig. 25, upper), not only are the pairs of vertical angles congruent, but because of pairings under the reflection, we can deduce that *all* the involved angles must be congruent. This situation does not occur in the other case (Fig. 25, lower).

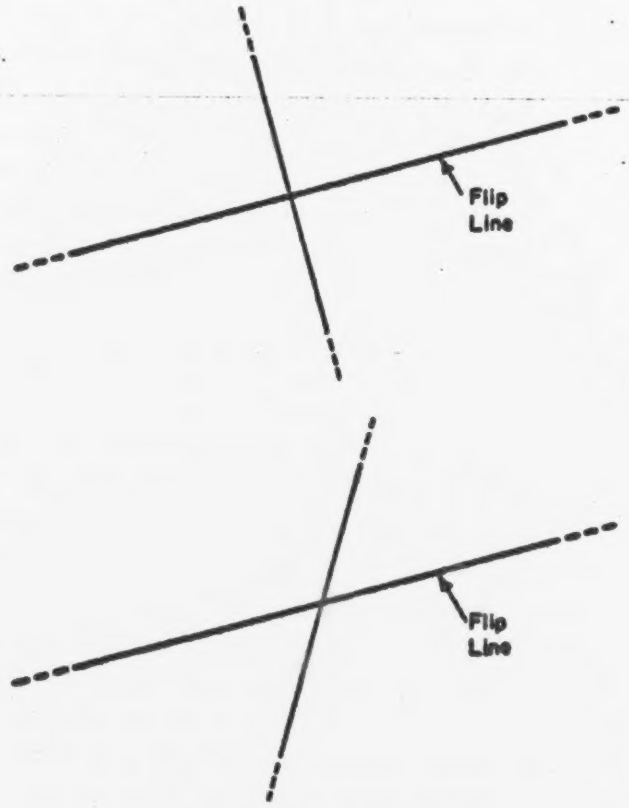


Figure 25

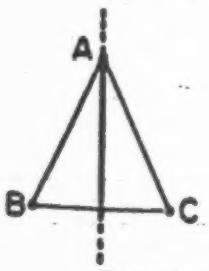
This observation motivates the following definition. Two lines are *perpendicular* whenever one is invariant under reflection about the other. (The word "two" is important here.) With this definition and an appropriate system of angle measure, one is ready to argue that perpendicular lines form four 90-degree angles.

Symmetry

Another example of the role of invariance is found in the study of symmetry.

A figure is said to have *line symmetry* whenever it is invariant under a reflection about some line. A triangle may have no line symmetries, one line symmetry, or three line symmetries. There are no triangles with exactly two line symmetries.

Consider any triangle with just one line of symmetry. Any such triangle must have one of its vertices on the symmetry line (see Fig. 26). Furthermore, its other vertices will be reflected images of each other (points B and C).



Triangle with one symmetry line.

FIGURE 26

Therefore the triangle has a pair of sides that are reflected images of each other, i.e., has two congruent sides (side $AB \cong$ side AC). The triangle also has a pair of angles that are reflected images of each other, so they are congruent ($\angle B \cong \angle C$). Since point B is the image of point C, line BC is invariant under the reflection, and the symmetry line bisects segment BC. This means the symmetry line is the perpendicular bisector of the side included between the congruent angles (angles B and C). The usual properties of medians, altitudes, and angle bisectors yield to this form of analysis equally well. (The term "isosceles" can be introduced at the appropriate time.)

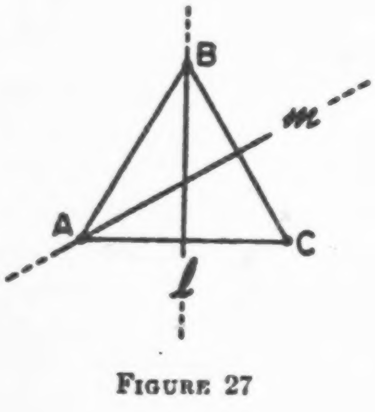


FIGURE 27

If a triangle has two line symmetries, ~~then it must have three. In Figure 27, lines~~ l and m are lines of symmetry; so side $AB \cong$ side AC, side $AB \cong$ side BC, and hence all three sides are congruent. That is, the triangle is equilateral. It can be shown that equilateral triangles have three line symmetries. In this case, the altitude from C lies along the third line of symmetry of the triangle.

Quadrilaterals

The study of quadrilaterals and their properties can parallel that of triangles, proceeding from quadrilaterals with no line symmetries to those with four lines of symmetry. One interesting difference between quadrilaterals and triangles is that the quadrilaterals can have symmetry lines which do not contain a vertex, while triangles cannot. As a result, we make a distinction between *diagonal symmetries* and *mediator symmetries*. A polygon has diagonal symmetry if the line containing one of its diagonals is a line of symmetry. It has mediator symmetry if one of its medians is a line of symmetry. (A median of a quadrilateral joins the midpoints of opposite sides of the figure.) Each of a quadrilateral's line symmetries will be either a diagonal symmetry or a mediator symmetry.

The reader may wish to verify that a quadrilateral with just one line of symmetry is either a kite or an isosceles trapezoid; with just two lines of symmetry is either a rectangle or a rhombus; with four lines of symmetry is a square. Just as we have shown in the case of isosceles and equilateral triangles, it is possible to derive the usual properties of the special quadrilaterals from their symmetry properties (see Fig. 28).

There is an important class of quadrilaterals that is not covered by the line-symmetry classification. These quadrilaterals are ordinarily not invariant under line reflection but are invariant under

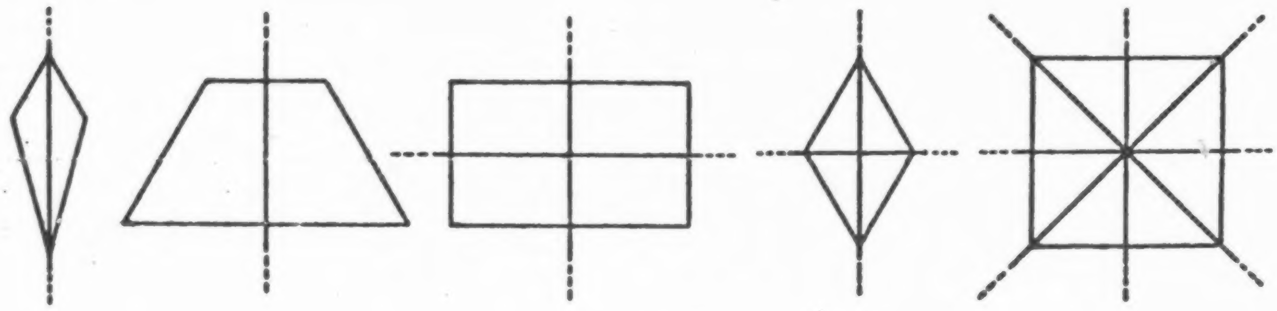


Figure 28

180-degree rotation. To have this invariance it is necessary for a quadrilateral to have its vertices arranged generally as shown in Figure 29. In this case A and C are images of each other, as are B and D. As was mentioned earlier, a 180-degree rotation maps a line onto a parallel line. So, in this case, since the rotation maps line AB onto line CD, we infer that segment AB and segment CD must be parallel. Similarly for segment BC and segment AD; and, of course, the segments in each of these pairs are congruent, because we have used another congruence mapping. For the same reasons, the diagonals of this quadrilateral bisect each other and opposite pairs of angles are congruent. These are the standard properties of *parallelograms.*

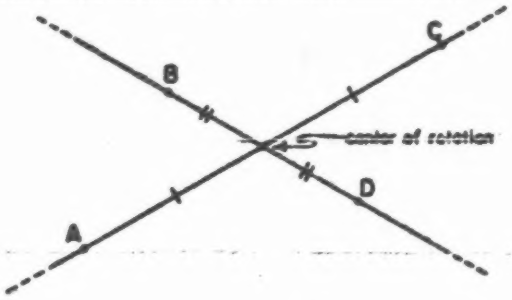


Figure 29

Constructions

An isometry-based context for geometry is rich with opportunities for students to discover construction techniques. As a simple example, take the problem of constructing a perpendicular to a line l through a point P not on line l. Students who have an understanding of the mappings discussed in this article have suggested the following methods:

1. Fold along line l and mark the point that corresponds with P under this fold. This point together with P determines the desired perpendicular.

2. Fold line l onto itself so that the fold line goes through point P. The crease determines the desired perpendicular.

3. Place a square-cornered card with one edge along line l and an adjacent edge running through P. Draw the desired line along the latter edge.

4. Use a flat, transparent, reflecting sheet (such as a stiff sheet of plastic or a thin piece of glass). Stand this sheet erect along line l, and mark the reflected image of point P in its apparent position "on the other side" of line l.

5. Again with a transparent reflecting sheet, stand the sheet erect so that its bottom edge determines a line through point P. Now rotate this sheet about, keeping the bottom on a line through P. As you do this you will find a position where the reflected image of the part of line l in front of the sheet coincides with

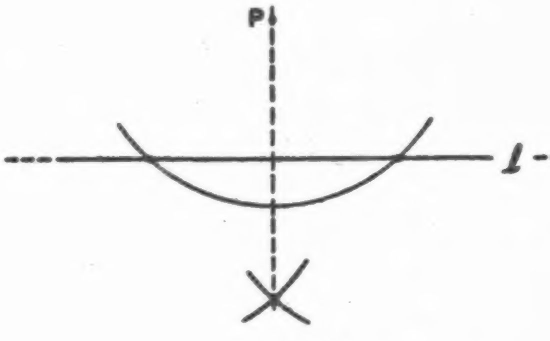


Figure 30

that part of l behind the sheet. In this position, mark the ends of the reflecting sheet. These points determine the desired line.

6. The usual compass construction (Fig. 30).

7. A variation on the usual construction technique (Fig. 31). Points X and Y are arbitrarily chosen.

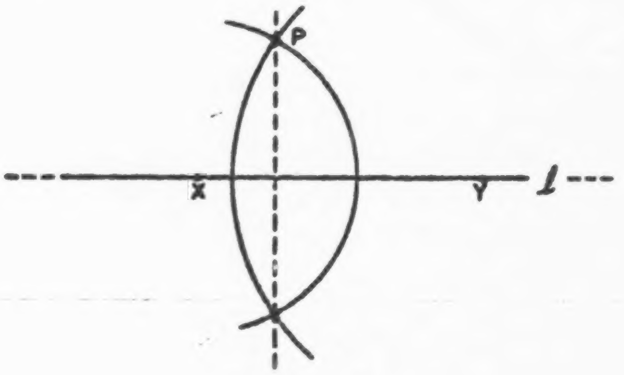


FIGURE 31

With this range of possibilities, the chances of each student's finding "his own" techniques are greatly increased. In addition, his knowledge of the mappings will promote more understanding of the various techniques than is achieved in the usual compass-straightedge context.

Although only one construction problem has been discussed in this paper, all of the standard plane-geometry constructions yield to attacks similar to those described above.

Summary

The isometry context, as described in this article, provides a wealth of opportunities to explore topics in geometry. This exploration is carried out in a setting that is accessible to students with a broad range of ability. Furthermore, innumerable opportunities arise for the student to gain experience in making arguments to verify or refute conjectures. The methods of attack presented here would not interfere with a later conventional (deductive) organization of geometry. Rather, they tend to strengthen the student's intuitive grasp of geometric facts so that he is better able to devote his attention to formal proofs and to appreciate the subtle relations among axioms, definitions, undefined terms, and theorems.

Two courses now available

The Secondary School Mathematics Curriculum Improvement Study has completed one year of experimental study of a new curriculum, Course I, intended for Grade 7. On the basis of this year of study, this course has now been published in revised form in two parts, each part approximately 375 pages. A Course II for Grade 8 has been published in an experimental version and consists of two parts, each approximately 350 pages. The textbooks for both of these courses are available in limited supply for review by interested persons. For information on cost and delivery dates write

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New York, New York 10027

SYLLABUS

E343

Teaching Mathematics in the Elementary Schools

E343

Teacher Competencies in Mathematics Methods for the Elementary School

The semester's work in E343 is planned and executed in such a way that every attempt is made to help teachers develop the following competencies for their work with children in the mathematics program:

1. The ability to use their knowledge and understanding of the real number system to provide a meaningful mathematics program for children.

2. The ability to use their knowledge and understanding of intuitive geometry to provide a more complete, meaningful mathematics program for children.

3. The ability to use their knowledge and understanding of the history and development of mathematics to provide a broad cultural foundation for a meaningful mathematics program for children.

4. The ability to provide meaningful experiences in mathematics for children, consistent with their knowledge and understanding of the social and quantitative aspects of their environment.

5. The ability to use their knowledge and understanding of the principles of child growth and development in planning and executing learning experiences in mathematics for children.

6. The ability to use their knowledge and understanding of the psychology of learning in planning and executing learning experiences in mathematics for children.

7. The ability to select content in mathematics for different grade levels consistent with the principles of child growth and development and research in the psychology of learnings.

8. The ability to use their knowledge and understanding of all areas of the curriculum in order to integrate and correlate these areas with mathematics.

9. The ability to examine and use past and present methods and techniques used in teaching mathematics in accordance with existing research.

10. The ability to use different kinds of instructional materials in providing learning experiences in mathematics for children.

11. The ability to make and use a variety of cooperative techniques for the purpose of evaluating individual and group performance in the mathematics program.

12. The ability to use professional books and periodicals to further professional growth in the area of mathematics.

FOREWORD

When constructing this guide, programs currently used in the public schools were the frames of reference for the outlines. No attempt was made to fit these outlines to any one methods textbook. As a consequence, some of the topics are treated in greater detail than the treatment furnished in the present textbook used in E343. Seldom will you find something included in the textbook but not included in the syllabus.

Using curriculum guides, pupils textbooks, scope and sequence charts, methods textbooks, and the experiences of mathematics education experts, the syllabus outlines were constructed. Obviously, the sequence of events in the outline do not exactly follow the present basic methods textbook. This is especially true in the sections dealing with number sentences, factors and multiples, and geometry. In several instances the ideas are widely separated in the book. In a few cases, the topics are treated under one heading in the syllabus and under two or more headings in the methods textbook.

Under existing planning and policies, mathematics content, as such, is left up to the mathematics department. The emphasis in the E343 syllabus is intended to be upon methods and materials. Again this is an area of disagreement between the methods textbook and the syllabus outlines. Most methods textbooks include a considerable amount of content, as such, prior to introduction of methods using that content. Our present textbook is no exception.

Persons using this syllabus should be aware of the differences between the content of the textbook and the program offered in E343. With this awareness, assignments will make sense when they are based upon the program as outlined by the syllabus rather than just the next 10 or 20 pages in the textbook. For certain topics supplementary reading will be necessary. Selected assignments from other resources will meet this need.

The present syllabus should be used as a guide in developing the scope and sequence of the program. It is not a perscription ~~no~~ any more than is the textbook. Rather it is a road map for the E343 course for both instructor and student.

CONTENT

E343

Teaching Mathematics in the Elementary School

	Topic	Page	Time in Days
Part I			
Chapter 1	Objectives and Philosophy	1	
Chapter 2	Planning and Organizing	2	(9)
Part II			
Chapter 3	Sets	3	
Chapter 4	Number Sentences	4	
Chapter 5	Factors and Multiples	5	(4)
	Test #1		(2)
Part III			
Chapter 6	Addition and Subtraction	6	
Chapter 7	Multiplication and Division	8	(10)
Part IV			
Chapter 8	Fractional Numbers	10	(8)
	Test #2		(2)
Part V			
Chapter 9	Geometry	13	(6)
Part IV			
Chapter 10	Verbal Problem Solving	15	(3)
Part VII			
Chapter 11	Evaluating and Planning	16	(4)
	Final Test		

PART I

OBJECTIVES AND PHILOSOPHY

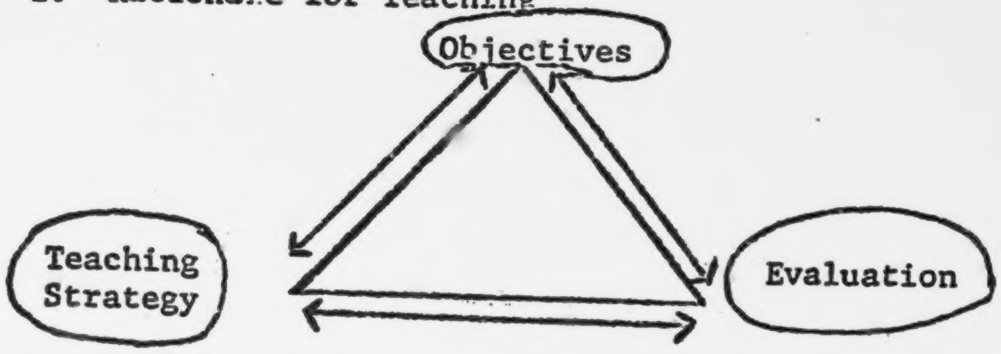
- I. Basic tenets of a modern mathematics program
  - A. General objectives of education
  - B. General objectives of mathematics education
  - C. Specific objectives of mathematics education
    - 1. Process goals
      - a. Acquire
      - b. Interpret
      - c. Evaluate
      - d. Communicate
    - 2. Behavioral goals

- II. Historical antecedents of current mathematics programs
  - A. Pre-history
  - B. Historical beginnings
    - 1. Roman-Greek period
    - 2. Mayan
  - C. Dark Ages
  - D. Renaissance
  - E. Colonial Period to 1821
  - F. 1821-1892
  - G. 1892-1935
  - H. 1935 - present

- III. Characteristics of modern programs in mathematics
  - A. Unifying themes of modern mathematics program
    - 1. Greater Cleveland Mathematics Program
    - 2. Madison Project
    - 3. SMSG
    - 4. Minnemath
    - 5. Stanford Project
    - 6. University of Illinois Arithmetic Project
  - B. Principles of learning used in modern mathematics programs

PLANNING AND ORGANIZING THE CLASSROOM FOR INSTRUCTION

I. Rationale for Teaching



II. Teaching Strategies
    A. Inquiry method
    B. Discovery method (inductive-deductive)
    C. Problem solving
    D. Expository method
    E. Spaced repetition
    F. Pupil involvement
    G. Homework
        1. Fallacy of typical homework
        2. Principles underlying the correct use of homework
        3. Examples of desirable homework assignments

III. Long range planning
    A. Scope and sequence
    B. Time allotments and pacing
    C. Unit planning without textbook
    D. Unit planning with textbook
    E. Planning for use of instructional resources

IV. Short range planning
    A. Steps in planning
        1. Objectives
        2. Readiness
        3. Approach
        4. Procedures
        5. Materials
        6. Summary
        7. Evaluation
    B. Initiating a new concept
    C. Practice and drill lessons
    D. Use of textbook and workbook
    E. Transitions from topic to topic and lesson to lesson

B

PART II

SETS

In each of the following sections emphasis first is placed upon the use of concrete materials for introduction of the concepts. The concepts are then further developed through pictorial and diagramatic set representations.

I. How to help children identify and describe sets.

II. Using equivalences to help children establish cardinal number names.

III. Using non-equivalence to establish the idea of ordinal number names.

IV. How to present the idea of the empty set.

V. How subsets are used to develop concepts related to operations with whole numbers.

VI. The importance of disjoint sets in fundamental operations.

VII. Relationships between set operations and operations on numbers.

VIII. How to present concepts of metric and non-metric geometry through the use of sets.

IX. Problem solving through the use of Venn diagrams.

NUMBER SENTENCES

I. Mathematics as a language

  A. Relationship to the English language

  B. Punctuation

  C. Precision of vocabulary

  D. Statements and Open Sentences: Their application and interpretation
    1. Frame arithmetic
    2. Simple and compound open sentences
    3. One variable and more than one variable
    4. Graphing solution sets on a number line

II. Using number sentences to solve verbal problems

  A. Translating verbal sentences and phrases into mathematical language.

  B. Expressing verbal problems as open sentences.

  C. Finding solution sets for equations and inequalities.

FACTORS AND MULTIPLES

I. How to develop meanings and interrelationships between factor, multiple, and divisor.

II. Helping Children Discover Prime Numbers, Composite Numbers, and the Role of One (Eratosthenes sieve, twin primes, etc.)

III. Discovering Divisibility Tests and Why They Work

IV. Methods of Factoring--Division, factor tree, etc.

V. Greatest Common Factor (Greatest Common Divisor)
   A. Determined through the use of sets (understanding)
   B. Determined through prime factorization (speed)
   C. Use of GCD in working with fractions--expressing in simplest form.

VI. Least Common Multiple
   A. Same as A above
   B. Same as B above
   C. Use of LCM in working with fractions--addition and subtraction of unlike fractions.

VII. Enrichment Activities
   A. Patterns
   B. Odd and Even Numbers
   C. Square and Triangular Numbers
   D. Exponents

PART III

ADDITION AND SUBTRACTION*

I. Relationship to counting
    A. Use of sets
    B. Use of number line

II. Basic facts for addition and subtraction
    A. Use of concrete materials--sets, examples of semi-concrete materials
    B. Inverse relationship--putting together and taking apart
    C. Writing the horizontal and vertical algorithms--use of frames
    D. How basic principles help us teach and learn basic facts (identity element, commutative principle, principle of one more, doubles, near doubles)
    E. Activities for developing automatic mastery

III. Addition
    A. Single-digit column addition
        1. Use of concrete materials--sets, examples of semi-concrete materials
        2. Horizontal and vertical algorithms--use of parentheses (associative principle)
        3. Teaching the unseen numeral
        4. Checking addition
    B. Multi-digit addition without regrouping
        1. Activities with concrete and semi-concrete materials to build understanding of place value--counters, place value charts, abacus, place value grids, pictorial and graphic representations.
        2. Relationship to basic facts
        3. Use of expanded notation
        4. Use of mature form of algorithms
    C. Multi-digit addition with regrouping
        1. Pictorial representations of sets of ten and sets of one
        2. Horizontal algorithm--expanded notation using principles
        3. Vertical algorithm--expanded notation (Heavy emphasis on place value charts)
        4. Activities to develop efficiency and speed in column addition
            a. Higher decade addition
            b. Addition of multiples of ten
    D. Additional ways to check column addition
        1. Check of nines
        2. Rule of compensation
        3. Regrouping addends

ADDITION AND SUBTRACTION
(Continued)

IV. Subtraction
  A. Types of subtraction situations
    1. How many left
    2. How many more are needed (look for missing addend)
    3. Comparison (find the difference between two numbers)
  B. Multi-digit subtraction without regrouping
    1. Activities with concrete and semi-concrete materials to build understanding of place-value--counters, place value charts, abacus, place value grids, pcitorial and graphic representations
    2. Relationship to basic facts
    3. Use of expanded notation
    4. Use of algorithms
  C. Multi-digit subtraction with regrouping
    1. Same as B. 1. above
    2. Methods of subtraction
      a. Decomposition
      b. Equal-additions
      c. Complementary
  D. Methods of checking subtraction
    1. Addition method
    2. Subtraction method
    3. Check of nines
    4. Approximation

V. Enrichment in addition and subtraction
    1. Games
    2. Magic Squares
    3. Cross Number puzzles
    4. Mental arithmetic
    5. Madison Project materials
    6. Row-Peterson booklets
    7. Operations using number bases other than ten

* Vocabulary stressed--addend, sum, missing addend, plus, minus, subtrahend, minuend, and remainder or difference.

MULTIPLICATION AND DIVISION*

- I. Relationship to addition and subtraction and ultimately to counting
  - A. Use of sets
    - 1. Multiplication
      - a. Repeated addition
      - b. Cartesian products
    - 2. Division
      - a. Partitive
      - b. Quotative
  - B. Use of number line

- II. Basic facts for multiplication and division
  - A. Use of concrete materials--sets, examples of semi-concrete materials
  - B. Inverse relationship--putting together and taking apart
  - C. Writing the horizontal and vertical algorithms--use of frames
  - D. How basic principles help us teach and learn basic facts (Identity element, commutative principle, distributive principle of multiplication over addition, principle of multiplying by zero, squaring)
  - E. Activities for developing automatic mastery

- III. Multiplication
  - A. Single-digit multiplier without regrouping
    - 1. Use of concrete materials--sets, examples of semi-concrete materials
    - 2. Relationship to basic facts
    - 3. Horizontal and vertical algorithms--use of parentheses (expanded notation using associative and commutative principles)
    - 4. Position of product numerals--extensive use of place value charts and counting men
    - 5. Checking multiplication--repeated addition
  - B. Single-digit multiplier with regrouping
    - 1. Activities with concrete and semi-concrete materials to build understanding of place value--counters, place value charts, abacus, place value grids, pictorial and graphic representations
    - 2. Use of expanded notation stressing the distributive principle
    - 3. Use of algorithms (mature form)
  - C. Multi-digit multiplier and multiplicand
    - 1. Use of arrays
    - 2. Expanded notation and use of distributive principle
    - 3. Placement of partial products
    - 4. Role of 0 in multiplier and multiplicand
    - 5. Use of algorithm (mature form)

MULTIPLICATION AND DIVISION
(Continued)

III. (Continued)
  D. Methods of checking mathematics
    1. Division
    2. Casting out nines
    3. Expanded notation
    4. Reversing factors

IV. Division
  A. Uneven Division--handling remainders
    1. As a whole number
    2. As a fraction
    3. As a rounding off process
  B. Multi-Digit Dividends with Single Digit Divisors in Division
    1. Use of arrary patterns
    2. Expanded notation
    3. Subtractive method
    4. Immature method of placement of quotient
    5. Mature method
    6. Checking division by repeated subtraction
  C. Multi-Digit Divisors in Division
    1. Use of array patterns
    2. Expanded notation
    3. Subtractive method--mature method
    4. Estimating quotients
      a. apparent method of trial divisors
      b. increase-by-one method of trial divisors
  D. Checking in Division
    1. Multiplication
    2. Approximation
    3. Check of nines

V. Enrichment in Multiplication and Division
    1. Doubling method
    2. Gelosia method
    3. Galley method
    4. Front-end method
    5. Egyptian division method
    6. Permutations and combinations
       (Arrangements and selections)
    7. Operations using bases other than ten

* Vocabulary stressed: factor, product, missing factor, times, multiply, divide, multiplicand, multiplier, partial product, product, divisor, dividend, quotient, and remainder.

PART IV

FRACTIONAL NUMBERS

- I. Foundational Program for Fractional Numbers
  - A. Different Situations in which Fractions Are Used
    - 1. To represent parts of a whole
    - 2. To represent one of a group of units
    - 3. To represent division of a whole number
    - 4. To represent a ratio
  - B. Developing the Concept of Fractional Numbers
    - 1. Use of concrete and semi-concrete materials--e.g., parts of real objects, number line representations, fraction charts, and fractional cutouts
    - 2. Representation of fractional numbers with emphasis placed upon the relationships which exist among them
      - a. Common fractions
      - b. Decimal fractions
      - c. Percents
    - 3. Ordering fractional numbers
    - 4. Learning to read and write fraction, decimal, and percent numerals
  - C. Equivalent Fractions
    - 1. Determine the meaning of equivalent fractions through the use of pictorial and graphic representations
    - 2. Use of factors, multiples, and the multiplicative identity to express fractional numbers in different terms

- II. Addition and Subtraction of Common Fractional Numbers
  - A. Foundation Activities
    - 1. Joining and separating congruent and discrete regions through the use of concrete objects
    - 2. Use of number lines and other graphic representations to solve addition and subtraction situations
    - 3. Relating the basic principles of whole numbers to fractional numbers--e.g., commutative, associative, etc.
  - B. Development of Formal Procedures in the Addition and Subtraction of Proper and Improper Fractions and Mixed Numbers
    - 1. Proper Fractions
      - a. Horizontal and vertical algorithms
        - (1) Like denominators
        - (2) Unlike denominators
      - b. Checking by pictorial and graphic representations and inverse relationships
      - c. Oral and mental activities
    - 2. Improper fractions and mixed numbers
      - a. Renaming improper fractions and mixed numbers
      - b. Horizontal and vertical algorithms
        - (1) Like deonominators
        - (2) Unlike deonominators

FRACTIONAL NUMBERS
(Continued)

- II. (Continued)
  - 2. (Continued)
    - c. Regrouping
    - d. Other numerical checking activities

- III. Multiplication and Division of Common Fractional Numbers
  - A. Foundational Activities
    - 1. Use of models to develop fractional multiplicative concepts
      - a. Pictorial representations
      - b. Unit regions
      - c. Number lines
    - 2. Use of models to develop fractional divisive concepts
      - a. Pictorial representations
      - b. Number line
    - 3. Determinational the relationship between division and multiplication of fractional numbers
    - 4. Determination of the basic principles which pertain to each of the two operations upon fractional numbers; i.e., closure, commutativity, associativity, multiplicative identity, distributivity, pm0, and reciprocal
  - B. Development of Formal Procedures in the Multiplication and Division of Fractional Numbers
    - 1. Multiplication
      - a. Basic horizontal algorithm used with common fractions
      - b. Variations involved when working with whole numbers and mixed numbers
        - (1) Use of vertical algorithm
        - (2) Expressing all factors in common fraction form
      - c. Renaming products
    - 2. Development of the concept of "Cancellation" through the use of renaming, the associative and commutative principles, and the multiplicative identity
    - 3. Division
      - a. Equal-denominator method
      - b. Inversion method
    - 4. Checking multiplication and division by pictorial and graphic representations and by inverse relationships

- IV. Decimals and Per Cents
  - A. Foundation Program. See Part I of Fractional Numbers
  - B. Situations which require the extension of place value and their interpretations through decimals and per cents
  - C. Operations with Decimal Fractions--Addition, Subtraction, Multiplication, and Division
    - 1. Relating the basic principles of fractional numbers to computations with decimal fractions.

FRACTIONAL NUMBERS
(Continued)

IV. (Continued)
  C. (Continued)
    2. Computation with decimal fractions through the use of common fractions, powers of ten, number lines, diagrams, algorithms, etc.
    3. Applications of decimal approximation of fractional numbers
    4. Applications of repeating decimals
  D. Rate-pair interpretation of per cent
    1. See 7.11e of Peterson and Hashisaki
    2. Approaching the solution of per cent problems through the use of proportions

V. Fractional numbers as ratios
  A. Identification of ratio and rate
  B. Use of equal ratios in equations

VI. Enrichment Activities
  A. Graphing on a number line
  B. Different number bases
  C. Historical development
  D. Puzzles and games

PART V

GEOMETRY

(Metric and Non-metric)

I. Introduction to Geometry in the Elementary School
  A. Overview
    1. Historical orientation
    2. Current trends
  B. Rationale for teaching geometry
    1. General objectives
    2. Specific objectives

II. Concept Development
  A. Intuitive Approach
  B. Environmental Orientation
  C. Teacher-pupil Dialogues

III. Essential Primitive Concepts
  A. Non-metric Geometry
  B. Metric Geometry

IV. Teaching of Geometric Terms and Symbols
  A. Perception Activities
  B. Vocabulary Activities
    1. Hearing
    2. Saying
    3. Seeing
    4. Writing
  C. Activities with Geometric Symbols

V. Teaching of Linear Figures and Linear Measurement
  A. Activities with representations of linear figures
  B. Elementary Concepts of congruence
    1. Slides-Translations
    2. Turns-Rotations
    3. Flips-Reflections
  C. Measurement activities with linear figures
    1. Length
    2. Angle
    3. Area
    4. Pythagorean Theorem
  D. Sketching of Linear figures and formal constructions of linear figures
  E. Classification of linear figures through set relationships
  F. Ordering by Size--primitive concept of comparison
  G. Similarity--Intuitive introductory activities

GEOMETRY

(Metric and Non-metric)
(Continued)

V. (Continued)
  H. Symmetry--Intuitive introductory activities
  I. Parallelism and Perpendicularity
    1. Paper folding and cutting activities
    2. Construction activities
  J. In depth approach to the sequence of geometric study

VI. Teaching of the Common Solids
  A. Activities with representations of common solids
  B. Measurement Activities with Common Solids
    1. Surface Areas
    2. Volumes
  C. Constructing models of solids
  D. Comparisons of solids

VII. Additional Measurement Topics
  A. English, Troy, Metric Systems
  B. Liquid and dry measurement
  C. Time
  D. Weight
  E. Temperature

VIII. Graphing
  A. Reading and constructing graphs
    1. Bar graphs
    2. Line graphs
    3. Pictorial graphs
    4. Circle graphs
  B. Graphing on a number line
  C. Cartesian coordinates
    1. Ordered pairs
    2. Plotting points
    3. Graphs of functions
    4. Graphing inequalities

XI. Enrichment Activities
  A. Geometric illusions
  B. Topology
  C. Paperfolding
  D. Mobiles
  E. Activities relating to the historical development of measurement; e.g. dramatizations and making models.

PART VI

VERBAL PROBLEM SOLVING

I. Definition of a problem
  A. Psychological
  B. Verbal mathematics problems

II. Purposes of problems
  A. Relationship to physical environment
  B. Relationship to mathematics program
    1. Practice
    2. Motivation
    3. Initiation

III. Issues in the teaching of problem solving
  A. Settings
  B. Factors associated with high achievement
  C. Reading skills related to problem solving
  D. Operations related to problem solving
  E. Readiness related to problem solving
  F. Procedures related to problem solving

IV. Types of Problems
  A. Single step
  B. Multi-step
  C. Non-numerical
  D. Insufficient data
  E. Superfluous data

V. Methods of Teaching Problem Solving
  A. Mathematical sentences
  B. Oral and mental problems
  C. Diagrams, graphs, drawings
  D. Restatement or analogies
  E. Pupil formulation of problems
  F. Too much, too little data
  G. Problems without numbers
  H. Set representations
  I. Logic
  J. Analysis
  K. Estimation

PART VII

EVALUATING AND PLANNING IN ELEMENTARY SCHOOL MATHEMATICS

I. Role of measurement in evaluation

II. Relationship of evaluation to objectives

III. Informal methods of evaluation
    A. Anecdotal record
    B. Rating scale
    C. Observation
    D. Interview

IV. Formal methods of evaluation
    A. Standardized tests
        1. Advantages and disadvantages
        2. Types available
            a. Diagnostic
            b. Inventory
            c. Achievement
            d. Attitudinal
            e. Meaning and Understanding
            f. Readiness
        3. Uses to be made of them
    B. Teacher-made tests
        1. Advantages and disadvantages
        2. Construction of items
            a. Interesting
            b. Varied
            c. Clear
        3. Uses to be made of them
    C. Textbook tests

V. Characteristics of a good test:
    A. Valid
    B. Reliable
    C. Proper format
    D. Easily scored

VI. Planning for individual differences
    A. Grouping for instruction
        1. Small group
            a. Ability
            b. Diagnosed need
            c. Interest
            d. Committee
        2. Whole group
            a. Levels of procedure
            b. Levels of content

VI. (Continued)
  B. Differentiation of assignments
    1. On the basis of ability
    2. On the basis of diagnosed need
    3. On the basis of interest

VII. Planning the learning environment
  A. Useful seating arrangements
  B. Useful traffic patterns
  C. Effective bulletin boards
  D. Mathematics corner
  E. Mathematics library